A 0-1 law for circle packings of coarsely hyperbolic metric spaces and applications to cusp excursion

Joint with Giulio Tiozzo

$\boxed{\text{PART I}}$

Khinchin's Theorem 1926 : $\psi : \mathbb{N} \to \mathbb{R}^+$

monotone decr (maybe other hypotheses).

Let $\Theta(\psi) := \{ x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{\psi(q)}{q}$ for ∞-ly many $\frac{p}{q} \in \mathbb{Q} \}$

Then

$$\sum_{q \in \mathbb{N}} \psi(q) = \infty \Rightarrow \Theta(\psi) \text{ has full measure}$$

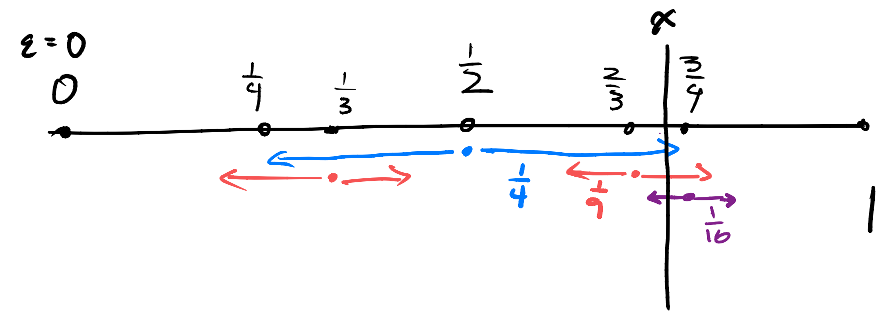
and

$$\sum_{q \in \mathbb{N}} \psi(q) < \infty \Rightarrow \Theta(\psi) \text{ has measure zero}$$

restrict to $[0,1]$ to get a "0-1" law

Application $\psi_\varepsilon(q) = \frac{1}{q^{1+\varepsilon}}$

$\Theta(\psi) = \{ x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$ for ∞-ly many $\frac{p}{q} \in \mathbb{Q} \}$



so for this x is in 3 of these balls. Will it be in ∞-ly many such balls?
If $\varepsilon > 0$ then the balls get even smaller

by Khinchin:

$$\sum_{q \in \mathbb{N}} \psi_\varepsilon(q) = \sum_{q \in \mathbb{N}} \frac{1}{q^{1+\varepsilon}} \begin{cases} = \infty & \text{for } \varepsilon = 0 \\ < \infty & \text{for } \varepsilon > 0 \end{cases}$$

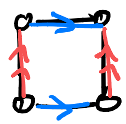
hence

$$\Theta(\psi_\varepsilon) \begin{cases} \text{has full measure for } \varepsilon = 0 \\ \text{has measure zero for } \varepsilon > 0. \end{cases}$$

In our picture, with probability 1, x is in ∞-ly many balls of radius $\frac{1}{q^2}$.

circle packings for the hyperbolic plane

by example

$\Sigma_{1,1}$ = 

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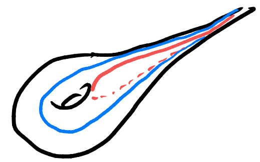
punctured torus

S = 

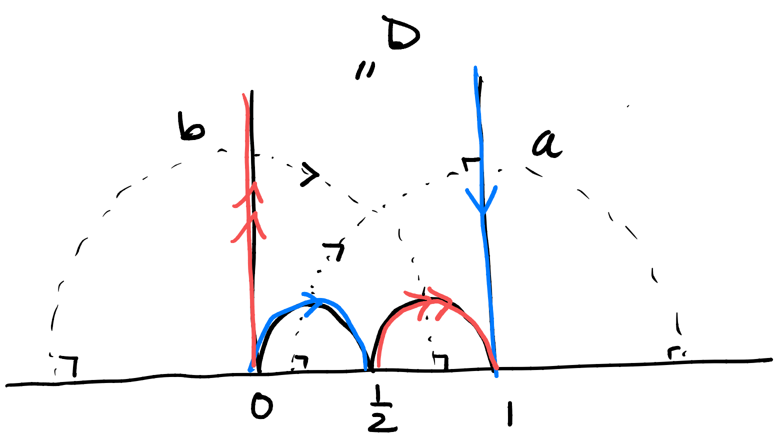
punctured torus with a metric

Let $S = \mathbb{H}^2/_{\Gamma} \cong \Sigma_{1,1}$ finite area

where $\pi_1(\Sigma_{1,1}) \cong F_2 \cong \Gamma < \mathrm{Isom}(\mathbb{H}^2)$

is a hyperbolic structure on $\Sigma_{1,1}$

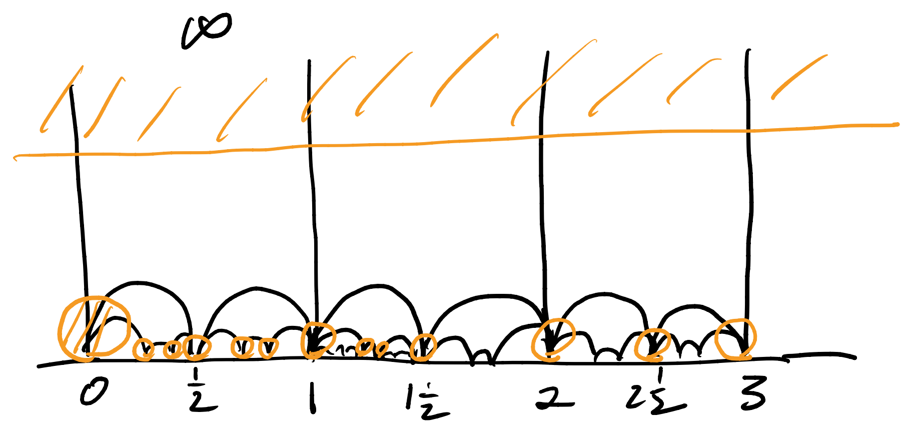
e.g. $\Gamma = \langle a, b \rangle$ where

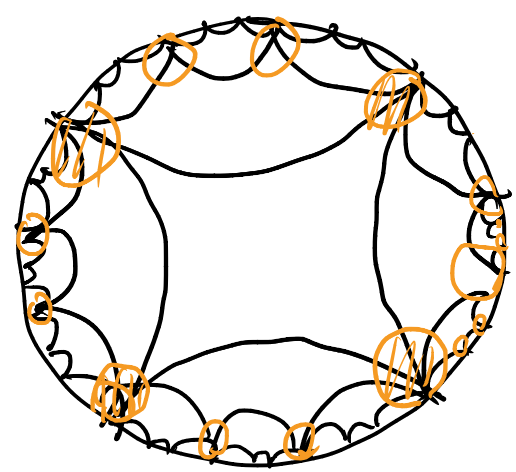


exercise: use that $g \in \mathrm{Isom}\,\mathbb{H}^2 = \{$Möbius transf. with coeffs in $\mathbb{R}\}$ is determined by the image of any 3 points in $\partial\mathbb{H}^2$ to identify the functions a, b.

tiling of $\mathbb{H}^2$ by $\Gamma . D$

⇝ circle packing (Γ-equivariant)

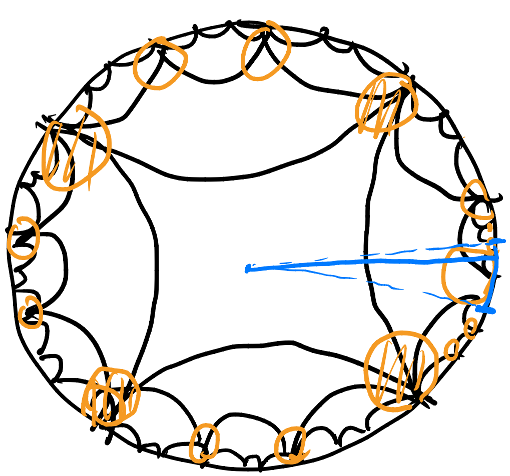




Defn: a circle in $\mathbb{H}^2$ tangent to $\partial \mathbb{H}^2$ is a horosphere. Its interior is a horoball. The point of tangency is the center of the horosphere/ball.

Let $\mathcal{P} = \{$centers of horoballs in the packing$\}$ and r_p = Euclidean radius of the horoball centered at p from a fixed original packing

Let $H_p(r)$ be a shadow of the horoball centered at p with radius r.



analogy to classical setting of Khinchin:

$\mathbb{R} \rightsquigarrow S^1$

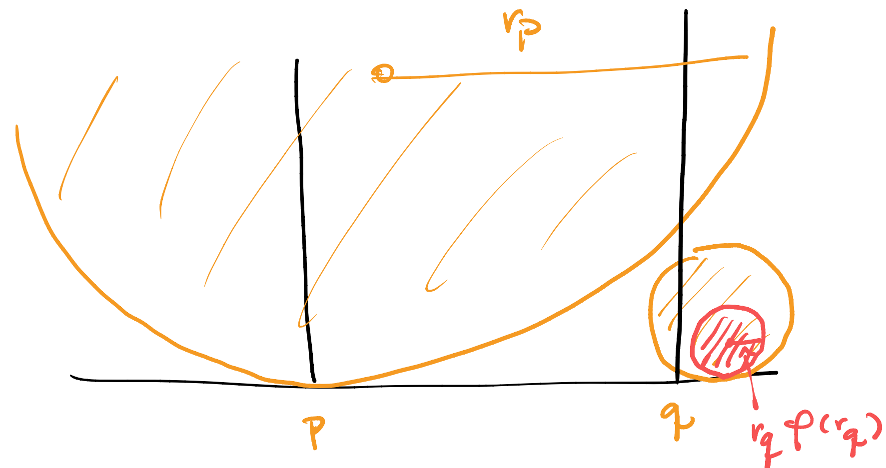
Lebesgue $\rightsquigarrow$ arclength

$\psi : \mathbb{N} \to \mathbb{R}^+$ decr $\rightsquigarrow$ $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ increasing

Rationals $\frac{p}{q}$ $\rightsquigarrow$ horoball centers $p \in \mathcal{P}$

$\{x : |x - \frac{p}{q}| < \frac{\psi(q)}{q} = \frac{1}{q}\psi(q)$ ∞-ly often$\}$ $\rightsquigarrow$ $\{x : x \in H_p(r_p \varphi(r_p))$ ∞-ly often$\}$

$= \Theta(\psi)$ $\qquad =: \Theta(\varphi)$



$\varphi \equiv 1$ $\Rightarrow$ no shrinking

in this case, a.e. x in ∞-ly many shadows.

$\varphi < 1$ $\Rightarrow$ shrinking, x may no longer be in ∞-ly many shadows

Thm: (Stratmann-Velani, Sullivan)

[Khinchin-type Theorem] for small $\lambda < 1$,

$$\sum_{n \in \mathbb{N}} \varphi(\lambda^n) < \infty \iff \Theta(\varphi) \text{ has measure zero}$$

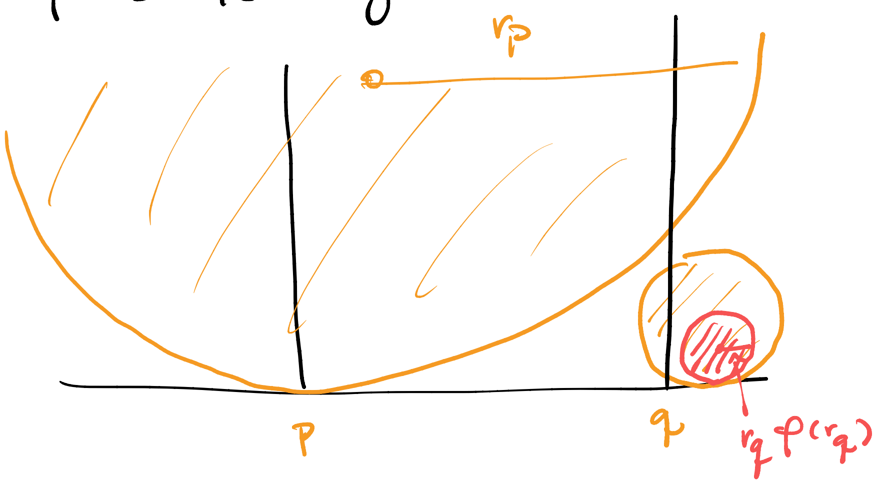
$$= \infty \iff \Theta(\varphi) \text{ has measure one}$$

Note: φ incr $\Rightarrow$ $\varphi(\lambda^n)$ decr. in n

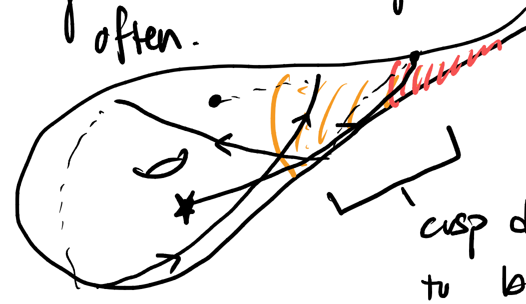
Application to cusp excursion

Horoball packing projects to neighborhood of the cusp.

φ shrinks neighborhoods.



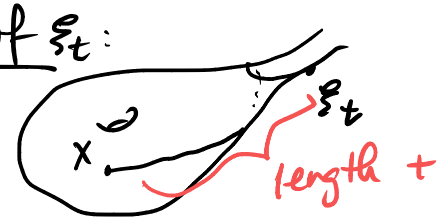
if they stay the "same size" ($\varphi \equiv 1$), a.e. geodesic visits neighborhood ∞-ly often.



cusp depth = distance to boundary of original horoball

Q: What is the optimal amount of shrinking?

defn of ξ_t:



Thm (Stratmann-Velani, Sullivan)

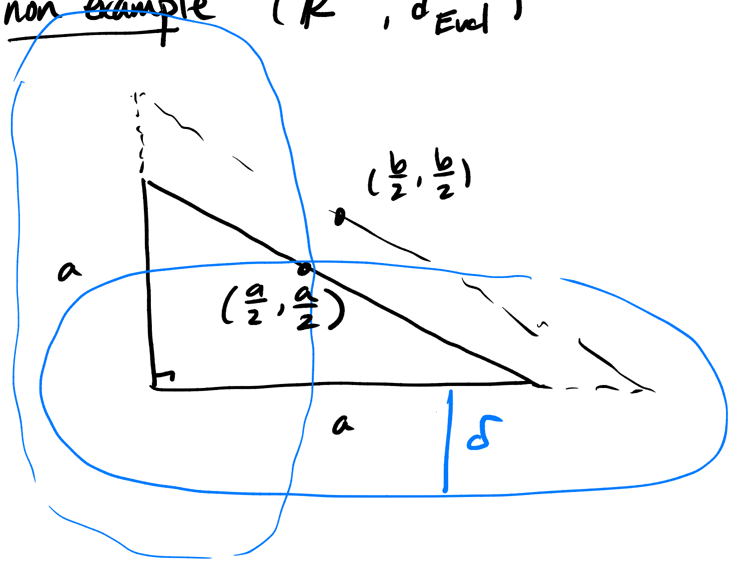
[Logarithm Law] For a.e. $\xi \in S^1$,

$$\limsup_{t \to \infty} \frac{\text{cusp depth}(\xi_t)}{\log(t)} = 1.$$

with Tiozzo, we generalize these to the setting of coarsely hyp. metric spaces which are geometrically finite. Will introduce these concepts now.

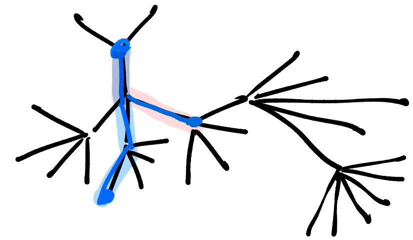
__Defn:__ (X,d) metric space is (Gromov) __hyperbolic__ if $\exists\, \delta > 0$ s.t. for any geodesic triangle, δ-nbhd of 2 edges contains the third.

__non example__ $(\mathbb{R}^2, d_{Eucl})$



δ depends on a

__example__ $X = T$ tree, d = path metric



can be ∞ valence!

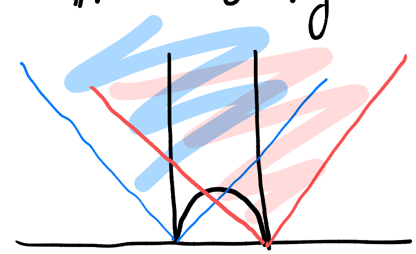
geodesic triangles are degenerate

Any $\delta > 0$ works.

"hyperbolic metric spaces are almost trees"

__exercise__

$\mathbb{H}^2$ $\delta = \log 2$



δ-nbhds in $\mathbb{H}^2$

$\boxed{\text{Hint:}}$ explain, and use again, this fact:

__Fact:__ since $PSL(2,\mathbb{R}) \cong \mathrm{Isom}\, \mathbb{H}^2$ is triply transitive on $\partial \mathbb{H}^2$, all ideal triangles are isometric example hyperbolic crochet

and MEGL "reverse escher" project

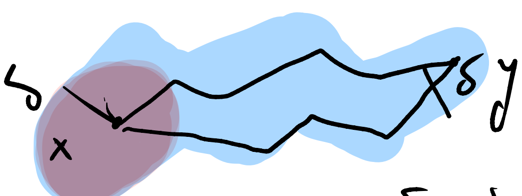
Defn: (X,d) is proper if closed metric balls are compact, and geodesic if $\forall\, x,y \in X$ $\exists$ geodesic x to y.

non-ex. of proper: infinite valence tree

$\boxed{\text{From now on, } (X,d) \text{ always proper geod. hyp metric space}}$

Fact: any 2 geodesics γ_1, γ_2 x to y are unif. bndd distance (dep only on δ)

pf:

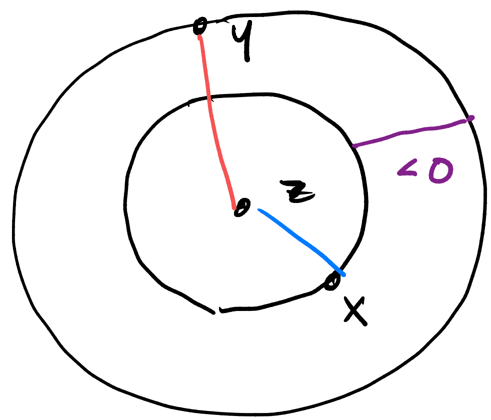


degenerate $\triangle$ vertices x, y, x. δ-nbhd of any 2 sides contains the third

Defn: Busemann function centered at z

$$\beta_z(x,y) = d(x,z) - d(y,z)$$

signed relative distance



exercise: $\beta_z(x,y) = -\beta_z(y,x)$ anti-symmetric

$$\beta_z(x,w) + \beta_z(w,y) = \beta_z(x,y) \quad \text{cocycle}$$

$g \in \mathrm{Isom}(X)$,

$$\beta_{gz}(gx,gy) = \beta_z(x,y) \quad \text{equivariance}$$

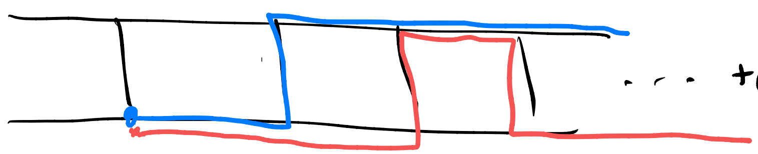
Defn: fix $o \in X$.

$$\partial X := \{\text{geodesic rays based at } o\} / \sim$$

where $\gamma_1 \sim \gamma_2$ if $\exists\, c$ s.t.

$$d(\gamma_1(t), \gamma_2(t)) \leq c \quad \forall t \geq 0.$$

e.g. Ladder graph



$$\partial X = \{+\infty, -\infty\}$$

e.g. $\partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\} = S^1$

Defn for $\xi \in \partial X$, define

$$\beta_\xi(x,y) = \liminf_{z \to \xi} \beta_z(x,y)$$

exercise: a) $\liminf = \limsup + O(\delta)$

b) $\beta_\xi(x,y) = -\beta_\xi(y,x) + O(\delta)$

c) $\beta_\xi(x,w) + \beta_\xi(y,w) = \beta_\xi(x,y) + O(\delta)$

quasi-anti-sym & quasi-cocycle

d) equivariance still true for $\xi \in \partial X$.

Defn:

Fix $o \in X$. A horosphere centered at $\xi \in \partial X$ of radius r is

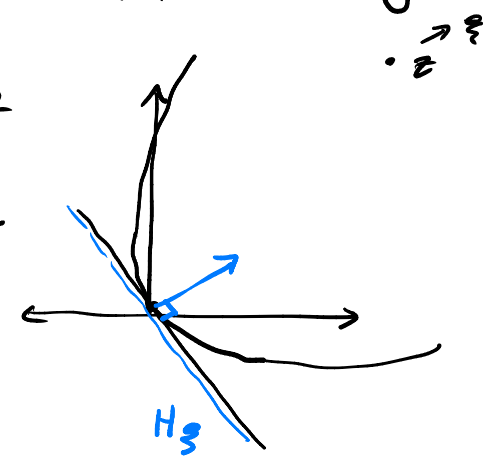
$$S_\xi = \{x \in X : \beta_\xi(x,o) = \log r\}$$

and horoball is

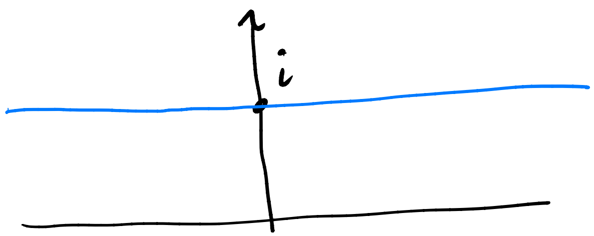
$$H_\xi = \{x \in X : \beta_\xi(x,o) \leq \log r\}$$

example $\mathbb{R}^2$

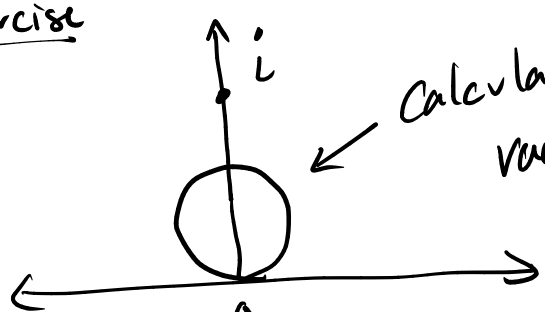
β_ξ is a limit (no $O(\delta)$)



example $\mathbb{H}^2$ β_ξ is a limit no $O(\delta)$



exercise



Calculate Eucl. radius of $H_0(r)$ with the basepoint i

exercise describe horospheres for ladder, $\mathrm{Cay}(\mathbb{F}_2)$

Def $H_\xi(r)$ = shadow of $H_\xi(r)$

$= \{\eta \in \partial X$ such that some geod. o to η intersects $H_\xi(r)\}$

Fact: $\{H_\xi(r) \mid \xi \in \partial X,\ r > 0\}$ generates the topology on ∂X.

exercise: the shadow topology on $\partial\mathbb{H}^2$ agrees with the usual topology on S^1.

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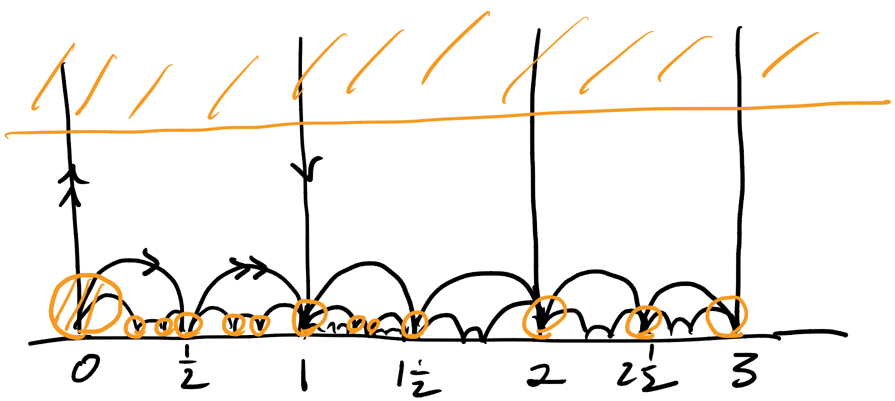
PART II

Last time

$S = \mathbb{H}^2/\Gamma \cong \Sigma_{1,1}$

$\Gamma < \mathrm{Isom}(\mathbb{H}^2)$

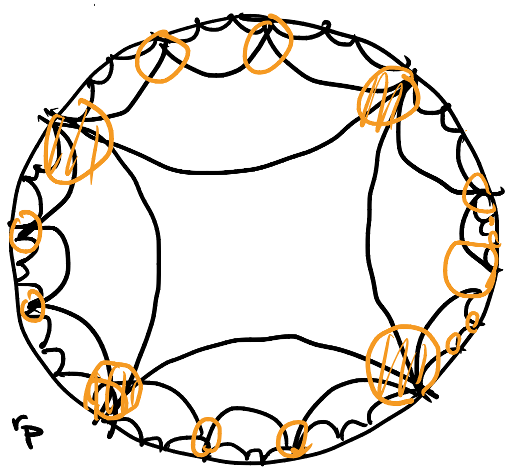


$\mathcal{P}$ = {vertices of ideal polygons in tilings}

"rational points" in $\partial\mathbb{H}^2$

$H_p(r)$ = horoball of Eucl. radius r at $p \in \mathcal{P}$

circle packing fix radii r_p



$\mathcal{H}_p(r)$ = shadow of $H_p(r) \subseteq \partial\mathbb{H}^2$

" neighborhood of p of radius r "

$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing

$\Theta(\varphi) = \{ \xi \in \partial\mathbb{H}^2 : \xi$ in ∞-ly many $\mathcal{H}_p(r_p \varphi(r_p)) \}$

Thm: (Stratmann-Velani, Sullivan)

[Khinchin-type Theorem] for small $\lambda < 1$,

$\sum_{n \in \mathbb{N}} \varphi(\lambda^n) < \infty \iff \Theta(\varphi)$ has measure zero

$= \infty \iff \Theta(\varphi)$ has measure one

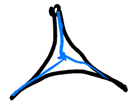
As an application,

Thm (Stratmann-Velani, Sullivan)

[Logarithm Law] For a.e. $\xi \in S^1$,

$$\limsup_{t \to \infty} \frac{\text{cusp depth }(\xi_t)}{\log(t)} = 1 .$$

Goal: generalize to hyp metric spaces

"tree like" triangles 

Fix (X,d) a proper geodesic hyp. metric space, $o \in X$.

$\partial X = \{\text{geod rays}\} / \text{bdd equiv}$

Recall for $\xi \in \partial X$, horoballs

$$H_\xi(r) = \{x \in X : \beta_\xi(x,o) \le \log r\}$$

"relative signed dist to ξ"

horospheres

$$S_\xi(r) = \{x \in X : \beta_\xi(x,o) = \log r\}$$

shadows

$$\mathcal{H}_\xi(r) = \{\xi \in \partial X : \exists \text{ geod. } o \text{ to } \xi \text{ which intersects } H_\xi(r)\}$$

generate the topology.

Exercise: topology independent of o

Thm (Bridson-Haefliger)

$\exists$ natural metric $d_{\partial X}$ and $\varepsilon > 0$ s.t. the radius of $\mathcal{H}_\xi(r)$ in $d_{\partial X}$ is $\asymp r^\varepsilon$ (mult. consts)

when $X =$ Poincaré disk, $d_{\partial X} \asymp$ arclength on S^1

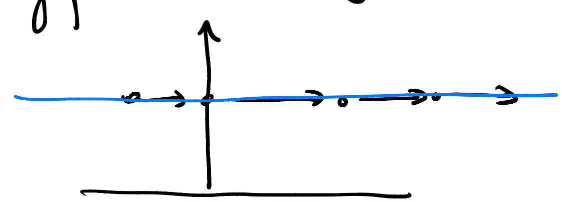
Isometries of X

Fact: $g \in \mathrm{Isom}\, X$. Then $g \in \mathrm{Homeo}(\partial X)$ and exactly one of the following occurs:

- g fixes a point in X (elliptic)
- g fixes ! point in ∂X (parabolic)
- g fixes exactly 2 pts in ∂X (loxodromic)

e.g. in $\mathbb{H}^2$

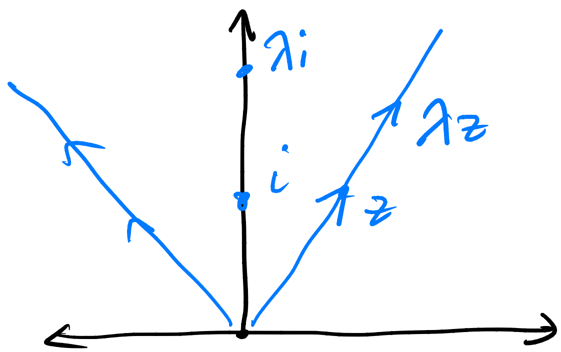
- g elliptic are rotations
- g parabolic conj. to $z \mapsto z+n$



$g\infty = \infty$! f.p.

in general $gp = p$ preserves horospheres centered at p

• g loxodromic conj to $z \mapsto \lambda z$



$g0 = 0$

$g\infty = \infty$

only fp's

in general:
g preserves the geodesic joining the fps and acts by translation along this geod, called the *axis of g*.

Defn: $\Gamma < \operatorname{Isom} X$ the limit set of Γ is

$$\Lambda_\Gamma := \overline{\Gamma o} / \Gamma o$$

Fact Λ_Γ independent of o

$$\Lambda_\Gamma \supseteq \{\text{fixed pts of elts of } \Gamma\}$$

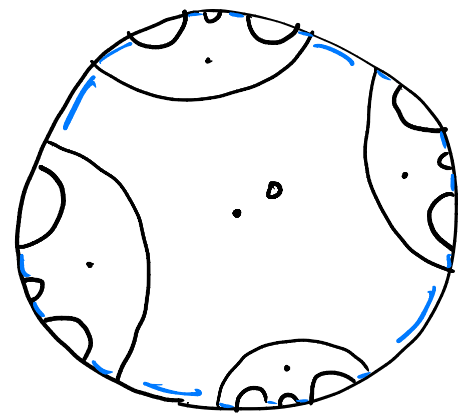
Defn: Γ nonelementary if $|\Lambda_\Gamma| \geq 3$

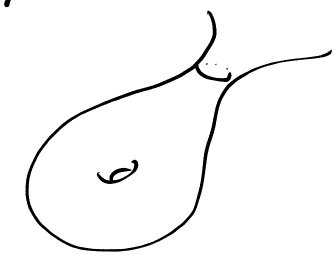
($\Rightarrow \Lambda_\Gamma$ uncountable)

Defn: C_Γ = convex hull of Λ_Γ.

Example $S = \mathbb{H}^2/\Gamma$ $\quad \Lambda_\Gamma = \partial\mathbb{H}^2 \quad C_\Gamma = \mathbb{H}^2$

Different metric: $S' = \mathbb{H}^2/\Gamma'$





note: S' has infinite area, but S was finite area.

by defn of fundamental domain, no accumulation of Γo in blue regions.

Λ_Γ = cantor set

Defn: $\xi \in \Lambda_\Gamma$ is *parabolic limit pt* if $\exists$ parabolic element fixing ξ.

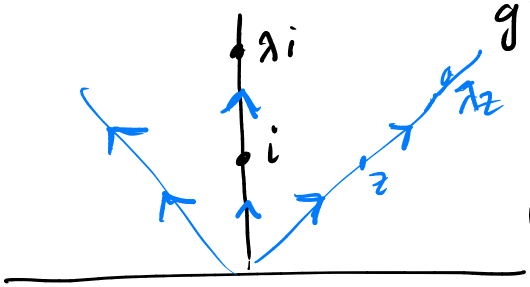
$\xi \in \Lambda_\Gamma$ is a *conical limit pt* if $\exists$ seq $z_n \to \xi$ bndd dist. from a geod.

Defn ctd

$p \in \Lambda_\Gamma$ parabolic is <u>bounded parabolic</u> if $\mathrm{Stab}_\Gamma(p) \curvearrowright \Lambda_\Gamma \setminus \{p\}$ is cocompact.

examples

- Any f.p. of loxodromic elt will be conical. In $\mathbb{H}^2$, picture is conjugate to



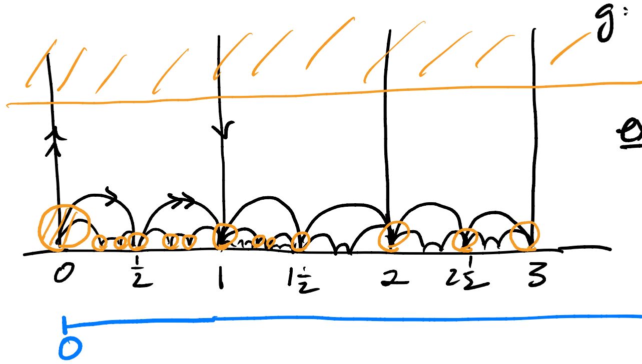
$g: z \mapsto \lambda z$

$\infty = g^+$ for $\lambda > 1$

$g^n i = \lambda^n i \to \infty$ and is on a geodesic so clearly bndd dist. from one

- $S = \mathbb{H}^2/\Gamma$ finite area

$\{\text{parabolics in } \Lambda_\Gamma\} = \infty$



$g: z \mapsto z+n$ fixes ∞

exercise: what is n

fund. domain for $g \curvearrowright \partial\mathbb{H}^2 \setminus \{\infty\}$

Since $\Lambda_\Gamma = \partial\mathbb{H}^2$, $g: z \mapsto z+n$ acts cocompactly on $\partial\mathbb{H}^2$ hence is a bounded parabolic.

Defn: $\Gamma < \mathrm{Isom}\, X$ acts <u>properly discontinuously</u> if $\forall\, K$ compact,

$$\{ g \in \Gamma : gK \cap K \neq \emptyset \} \text{ is finite.}$$

Γ properly discts, non-elem <u>geometrically finite</u> if every pt in Λ_Γ is either conical or bndd parabolic.

Fact: if $\mathbb{H}^2/\Gamma$ finite area then Γ is geom. finite.

non-example

$S'' = \mathbb{H}^2/\Gamma'' =$ "flute surface"

Fact (Bowditch) If $\Gamma \curvearrowright X$ geom. finite then there are finitely many conjugacy classes of parabolics.

($\Rightarrow$ finite # "cusps" in the quotient)
(projection of parab. pts)

and $\mathcal{P} := \{\text{parab. pts}\}$ is countable.

(this fact $\Rightarrow$ flute surface is not geom. fin.)

Defn: A horoball packing $\{H_p(r_p)\}_{p \in \mathcal{P}}$ is quasi-Γ-invariant if $\exists$ c st. $\forall$ $p \in \mathcal{P}$, $g \in \Gamma$, the Gromov-Hausdorff distance btwn $g H_p(r_p)$ and $H_{gp}(r_{gp})$ is bndd by c.

Horoball packing is invariant if $c = 0$.

Prop: (Bowditch, B.-Tiozzo)
$\Gamma \curvearrowright X$ geom. fin. $\Rightarrow$ $\exists$ quasi-Γ-inv. horoball packing of X, and $\exists$ $\kappa > 0$ s.t.

$$\underbrace{\bigcup_{\gamma \in \Gamma} B(\gamma o, \kappa) \supseteq C_\Gamma - \bigcup \underbrace{\text{Horoballs}}_{\text{"cuspidal part"}}}_{\text{"non cuspidal part"}}.$$

Defn: $\Pi < \Gamma$ parabolic subgp if $\Pi = \mathrm{Stab}_\Gamma(p)$ some parabolic $p \in \Lambda_\Gamma$.

Π has mixed exponential growth if $\exists$ $a, b \geq 0$ s.t. for $t \geq 0$,

$$\#\{g \in \Pi : d(o, go) \leq t\} \asymp e^{bt}(t+1)^a$$

- $\asymp$ means up to unif mult. constants indep of t
- $(t+1)^a$ instead of t^a b/c $g = id$ always possible so LHS ≥ 1. need RHS bndd away from 0 to get unif. mult. constants.

Defn for $\Gamma' < \Gamma$ let

$$\delta_{\Gamma'} = \limsup \frac{1}{t} \log \#\{g \in \Gamma' : d(o, go) \le t\}$$

$$a_{\Gamma'} = \limsup \frac{\log \#\{g \in \Gamma' : d(o,go) \le t\} - \delta_{\Gamma'} t}{\log t}$$

exercise: if Π has mixed exp growth $\asymp e^{bt}(t+1)^a$ then $\delta_\Pi = b$, $a_\Pi = a$.

exercise: $\delta_{\Gamma'}$, $a_{\Gamma'}$ indep of o

hence $\Pi' = g \Pi g^{-1} \Rightarrow \delta_{\Pi'} = \delta_\Pi$. $a_{\Pi'} = a_\Pi$

examples

- Fact: $S = \mathbb{H}^2/\Gamma$, δ_Γ = Hausdorff dim of Λ_Γ

so if area$(S) < \infty$ $\delta_\Gamma = 1$

- g loxodromic $\in$ Isom $\mathbb{H}^2$

$\Gamma = \langle g \rangle$ choose o on axis of g

so $d(o, g^n o) = n\, d(o, go)$

hence $\#\{n : d(o, g^n o) \le t\} = \left\lfloor \frac{t}{d(o,go)} \right\rfloor \le t$

then $\delta_\Gamma \le \limsup \frac{1}{t} \log(t) = 0$.

exercise $a_\Gamma = 1$.

- $\Pi = \langle p \rangle$ $p = z \mapsto z+1$

Fact: $A \in PSL_2\mathbb{R}$, $d_{\mathbb{H}^2}(i, Ai) = \log \frac{\sigma_1(A)}{\sigma_2(A)}$

σ_i singular values of A

idea of proof $A = U \Sigma V$, U, V unitary fix i

$\Sigma = \begin{pmatrix} \sigma_1 & \\ & \sigma_2 \end{pmatrix}$ and $d_{\mathbb{H}^2}(i, \Sigma i) = \log \frac{\sigma_1}{\sigma_2}$

exercise $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $\log \frac{\sigma_1(A^n)}{\sigma_2(A^n)} \sim \log n^2$ additive constants

then $\#\{g \in \Pi : d(i, gi) \le t\}$

$\asymp \#\{n : n \le e^{t/2}\} \asymp e^{t/2}$ mult. const. fine here

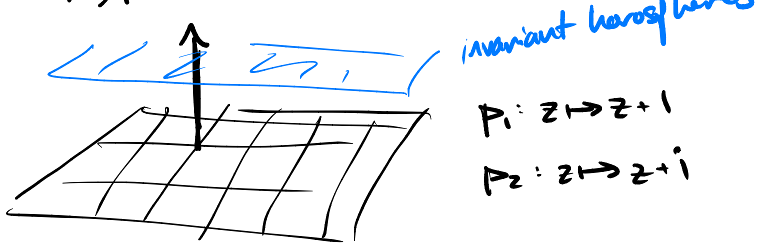
so $\delta_\Pi = \frac{1}{2}$.

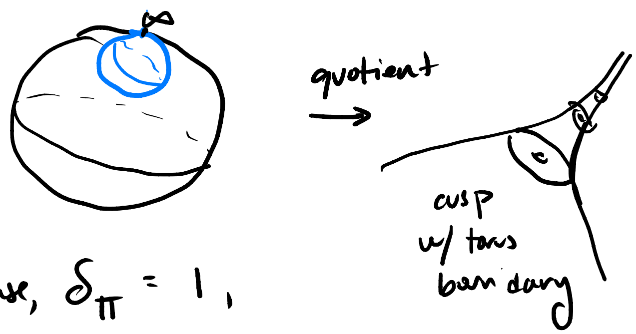
exercise $a_\Pi = 0$

• in $\mathbb{H}^3$, $\Pi = \langle p \rangle$ p parab. same.

Can also have

$\Pi = \langle p_1, p_2 \rangle \cong \mathbb{Z}^2$ "rank 2"





In this case, $\delta_\Pi = 1$,

$a_\Pi = 0$.

• in $\mathbb{H}^n$, Π parabolic $\Rightarrow$ $\Pi \underset{\text{virt.}}{\cong} \mathbb{Z}^k$

for some k, ie Π rank k abelian group.

Then $\delta_\Pi = \frac{k}{2}$.

($a_\Pi = 0$ again)

Khinchin-type Theorem

Assume only 1 cusp for simplicity

Assume $0 < \delta_\Pi < \delta_\Gamma$.

__Defn__: $\varphi: \mathbb{R}^+ \to (0,1]$ __Khinchin function__

if φ incr and $\exists$ $b_1 < 1$, $b_2 > 0$

such that

$$\varphi(b_1 x) \geq b_2 \varphi(x)$$

$\forall x \in \mathbb{R}^+$ (important only for small x)

__exercise__ $\beta > 0$ $\varphi(x) = \min\{\log(x^{-1})^{-\beta}, 1\}$

is a Khinchine function.

__Fix__ $\{H_p(r_p)\}_{p \in \mathcal{P}}$ quasi-Γ-invariant

horoball packing of X from Bowditch

__Defn__: for fixed $\lambda < 1$ let

$$S_n^\lambda(\varphi) := \bigcup_{\lambda^{n+1} \leq r_p < \lambda^n} \mathcal{H}_p(r_p \varphi(r_p)) \subseteq \partial X.$$

Defn: μ measure on ∂X is quasi-indep if $\exists C, \lambda$ s.t.

$$\mu(S_n \cap S_m) \leq C\mu(S_n)\mu(S_m).$$

Notation

$$\Theta_\lambda(\varphi) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \bigcup_{\underbrace{\lambda^{n+1}\leq r_p \leq \lambda^n}} H_p(r_p\varphi(r_p))$$

original targets abt the same size

$$= \limsup_{n\to\infty} S_n^\lambda(\varphi)$$

$$= \{x\in\Lambda_\Gamma : x\in H_p(r_p\varphi(r_p)) \text{ } \infty\text{-ly often}\}$$

Khinchin series

$$K_\lambda(\varphi) = \sum \varphi(\lambda^n)^{2(\delta-\delta_\pi)} (-2\log\varphi(\lambda^n)+1)^{a_\pi}$$

Notice for $S = \mathbb{H}^2/\Gamma$ finite area,

$$\delta_\Gamma = 1, \quad \delta_\pi = \frac{1}{2}, \quad a_\pi = 0$$

hence $K_\lambda(\varphi) = \sum \varphi(\lambda^n)$.

Thm (B.-Tiozzo) [Khinchin-type theorem]

$\exists$ measure μ proba on Λ_Γ ergodic wrt $\Gamma \curvearrowright \partial X$ and quasi-independent with nice scaling properties.

for any Khinchine function φ,

(1) $\mu(\Theta_\lambda(\varphi)) = 0$ if $K_\lambda(\varphi) < \infty$

(2) $\mu(\Theta_\lambda(\varphi)) = 1$ if $K_\lambda(\varphi) = \infty$.

Thm (B.-Tiozzo) [Logarithm law] Same μ.

For μ-a.e. $\xi\in\Lambda_\Gamma$,

$$\limsup_{t\to\infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \frac{1}{2(\delta_\Gamma - \delta_\pi)}.$$

Note: $d(\xi_t, \Gamma o)$ = "cusp depth"

recall $\bigcup B(\gamma o, k) \supseteq X - \bigcup H_p(r_p)$

A 0-1 law for circle packings of coarsely hyperbolic metric spaces and applications to cusp excursion

Joint with Giulio Tiozzo

PART III

Fix: (X,d) proper hyp. metric space.

$\Gamma < \mathrm{Isom}\, X$ geometrically finite.

For simplicity,

Assume X/Γ has one cusp i.e. up to conjugation, only one parabolic subgroup $\Pi < \Gamma$.

Assume Π has mixed exponential growth,

i.e.

$$\{g \in \Pi : d(o,go) \le t\} \asymp e^{\delta_\Pi t}(t+1)^{a_\Pi}$$

and $0 < \delta_\Pi < \delta_\Gamma$

(Rem: δ_Γ is called the critical exponent of Γ)

Recall / Def Λ_Γ = limit set of $\Gamma = \overline{\Gamma o} \setminus \Gamma o$

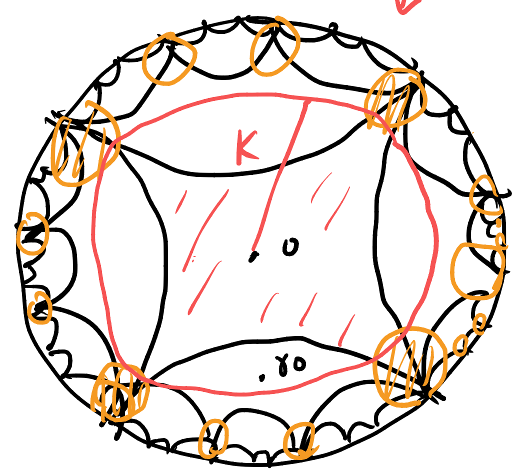
smallest closed Γ-inv. set in $X \cup \partial X$.

C_Γ = convex hull of Λ_Γ in X.

Fix $\{H_p(r_p)\}_{p\in\mathcal{P}}$ quasi-Γ-invariant horoball packing of X with

$$\bigcup_{\gamma\in\Gamma} B(\gamma o, K) \supseteq C_\Gamma - \bigcup_{\gamma} H_p(r_p)$$

(Γ acts cocompactly on $C_\Gamma - \bigcup_{\gamma} H_p(r_p)$)



Defn: for fixed $\lambda < 1$ let

$$S_n^\lambda(\varphi) := \bigcup_{\lambda^{n+1} \le r_p \le \lambda^n} \mathcal{H}_p(r_p \varphi(r_p)) \subseteq \partial X$$

Defn: μ measure on ∂X is <u>quasi-indep</u> if $\exists C, \lambda$ s.t. $\forall n, m$,

$$\mu(S_n^\lambda \varphi \cap S_m^\lambda \varphi) \le C \mu(S_n^\lambda \varphi)\mu(S_m^\lambda \varphi).$$

<u>Notation</u>

$$\Theta_\lambda(\varphi) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} \bigcup_{\underbrace{\lambda^{n+1} \le r_p \le \lambda^n}} H_p(r_p \varphi(r_p))$$

original targets abt the same size

$$= \limsup_{n \to \infty} S_n^\lambda(\varphi)$$

$$= \{ x \in \Lambda_\Gamma : x \in H_p(r_p \varphi(r_p)) \ \infty\text{-ly often} \}$$

<u>Khinchin series</u>

$$K_\lambda(\varphi) = \sum \varphi(\lambda^n)^{2(\delta - \delta_\pi)} (-2\log \varphi(\lambda^n) + 1)^{a_\pi}$$

<u>Notice</u> for $S = \mathbb{H}^2/\Gamma$ finite area,

$$\delta_\Gamma = 1, \quad \delta_\pi = \tfrac{1}{2}, \quad a_\pi = 0$$

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<u>Thm</u> (B.-Tiozzo)

[Khinchin-type Theorem]

$\exists$ measure μ proba on Λ_Γ ergodic wrt $\Gamma \curvearrowright \partial X$ and quasi-independent with nice scaling properties.

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<u>Thm</u> (B.-Tiozzo)

[Logarithm law] Same μ.

For μ-a.e. $\xi \in \Lambda_\Gamma$,

$$\limsup_{t \to \infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \frac{1}{2(\delta_\Gamma - \delta_\pi)}.$$

<u>Note:</u> $d(\xi_t, \Gamma o)$ = "cusp depth"

recall $\bigcup B(\gamma o, k) \supseteq C_\Gamma - \bigcup H_p(r_p)$

Pf of logarithm law

$$\varphi_\varepsilon(x) := \log(x^{-1})^{-\frac{1+\varepsilon}{2(\delta-\delta_\pi)}}$$

is a Khinchin function by prior exercise, and

$$K_\lambda(\varphi_\varepsilon) = \sum \varphi_\varepsilon(\lambda^n)^{2(\delta-\delta_\pi)} (-2\log\varphi_\varepsilon(\lambda^n))^{a_\pi}$$

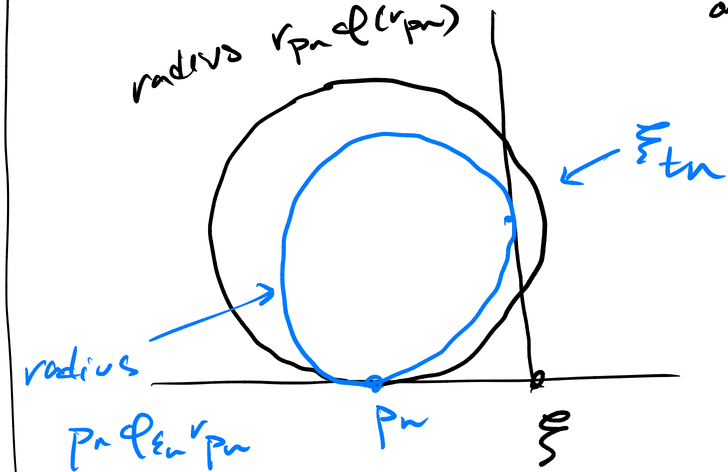
$$= \sum \log(\lambda^{-n})^{-1+\varepsilon} \cdot (-2)^{a_\pi}\left(\log\left((\log \lambda^{-n})^{1+\varepsilon/2(\delta-\delta_\pi)+1}\right)\right)^{a_\pi}$$

$$\asymp \sum \frac{1}{n^{1+\varepsilon}} \log(n+1)^{a_\pi} \qquad a_\pi \geq 0$$

calculus exercise ↑ diverges if $\varepsilon = 0$ and converges if $\varepsilon > 0$.

Then μ a.e. $\xi \in \Theta_\lambda(\varphi_0)$, choose maximal seq. $p_n \in P$ so that geodesic $[o,\xi)$ ~~passes~~ passes through $H_{p_n}(r_{p_n}\varphi_0 r_{p_n})$ in order, and $r_{p_n}\varphi_0 r_{p_n}$ monotone decr.

then, $\exists\, \varepsilon_n > 0$ s.t. $[o,\xi)$ tangent to $H_{p_n}(r_{p_n}\varphi_{\varepsilon_n} r_{p_n})$. ξ_{t_n} pt of tangency on $[o,\xi)$.



claim:

$$\limsup_{t\to+\infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \limsup_{n\to\infty} \frac{d(\xi_{t_n}, \Gamma o)}{\log t_n}$$

idea: $d(\xi_{t_n}, \Gamma o) \approx$ distance between horospheres maximized at $t = t_n$

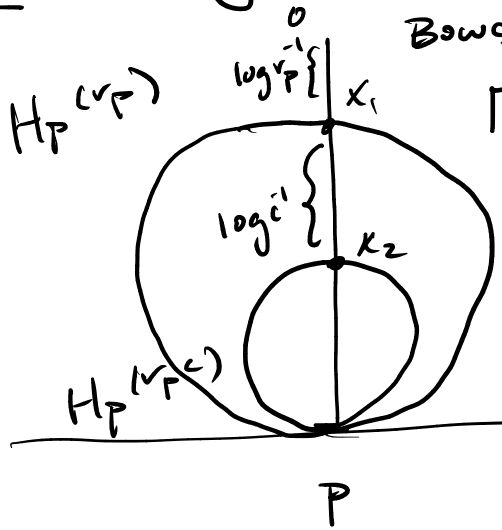
$\log t$ monotone incr. there is more...

Lemma 1:

up to additive constant

$$d(\xi_{t_n}, \Gamma_0) \sim -\log(\varphi_{\varepsilon_n}(r_{p_n}))$$

idea: for any $c<1$



Bowditch prop: Γ_0 bndd dist from $H_p(r_p)$.

How much further to get to $H_p(r_p c)$?

$\beta_p(0,x_1) = -\log r_p$

$\beta_p(0,x_2) = -\log r_p c$

intuition: horospheres are "balls centered at p", x_1, x_2 colinear on geod to p

cocycle property

$$\beta_p(x_1,0) + \beta_p(0,x_2) = \beta_p(x_1,x_2) = d(x_1,x_2)$$

"

$\log r_p - \log r_p - \log c$

$= -\log c$.

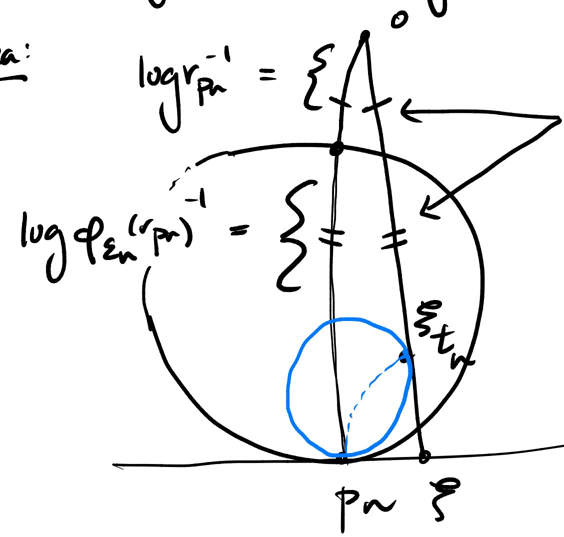
Now take $c = \varphi_\varepsilon r_{p_n}$ □

Lemma 2:

additive consts

$$t_n + \log \varphi_{\varepsilon_n}(r_{p_n}) \lesssim \log r_{p_n}^{-1} \lesssim t_n$$

idea:



hyp metric space lemma:

ξ_{t_n} = closest point projection of p_n to $[0,\xi)$

= bndd dist from inner triangle □

$$\overset{(L1)}{\Longrightarrow} \quad d(\xi_{t_n}, \Gamma_0) \sim -\log \varphi_{\varepsilon_n} r_{p_n}$$

$$= -\log\left(\left(\log r_{p_n}^{-1}\right)^{-(1+\varepsilon_n)/2(\delta_\Gamma-\delta_\Pi)}\right)$$

$$= \frac{1+\varepsilon_n}{2(\delta_\Gamma-\delta_\Pi)} \log\log r_{p_n}^{-1}$$

$$\le \frac{1+\varepsilon_n}{2(\delta_\Gamma-\delta_\Pi)} \log t_n \quad (L2)$$

$$\Longrightarrow \quad \limsup \le \frac{1+\varepsilon_n}{2(\delta_\Gamma-\delta_\Pi)} \quad \forall n.$$

If $\varepsilon_n > \varepsilon > 0$ some ε, then

$\xi \in \Theta_\lambda(\varphi_\varepsilon)$. But $\mu(\Theta_\lambda(\varphi_\varepsilon)) = 0$,

so we can choose ξ s.t.

$\xi \in \Theta_\lambda(\varphi_0)$ but $\xi \notin \Theta_\lambda(\varphi_\varepsilon)$

$\forall \varepsilon > 0$. Thus, $\varepsilon_n \to 0$ and the upper bound follows.

Lower bound

$$d(\xi_{t_n}, \Gamma o) \sim -\log \varphi_{\varepsilon_n} r_{p_n}$$

$$= \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log \log r_{p_n}^{-1}$$

(L2) lower $\gtrsim \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \left(\log \left(t_n - \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log\log(r_{p_n}^{-1})\right)\right)$

(L2) upper $\gtrsim \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log\left(t_n - \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log(t_n)\right)$

$\Rightarrow \limsup \geq \frac{1}{2(\delta_\Gamma - \delta_\Pi)}$ since $t_n \to \infty$, $\varepsilon_n \to 0$

$\square$

Thm (B.–Tiozzo)

[Khinchin-type theorem]

(A) $\exists$ measure μ proba on Λ_Γ ergodic wrt $\Gamma \curvearrowright \partial X$ and quasi-independent with nice scaling properties

(B) for any Khinchine function φ,

(1) $\mu(\Theta_\lambda(\varphi)) = 0$ if $K_\lambda(\varphi) < \infty$

(2) $\mu(\Theta_\lambda(\varphi)) = 1$ if $K_\lambda(\varphi) = \infty$.

We will prove (B) given (A)

Lemma 3 (Borel–Cantelli)

$(Y, \mathbb{P})$ measure space $A_n \subseteq Y$ measurable

(1) $\sum \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup A_n) = 0$

(2) $\sum \mathbb{P}(A_n) = \infty$ and $\exists c > 0$ s.t. $\mathbb{P}(A_n \cap A_m) \leq c\,\mathbb{P}(A_n)\mathbb{P}(A_m)$

$\Rightarrow \mathbb{P}(\limsup A_n) > 0$.

harder exercise or just find & read proofs

Lemma 4

$$\mu(S_n^\lambda \varphi) \asymp \varphi(\lambda^n)^{2(\delta-\delta_\Pi)} (-2\log \varphi(\lambda^n))^{a_\Pi}$$

idea: $\mu(S_n^\lambda \varphi) \asymp \# S_n^\lambda \varphi \, \mu(H)$

for any fixed $H \in S_n^\lambda \varphi$ by quasi-disjointness (Dirichlet-type thm) of elts of $S_n^\lambda \varphi$ and scaling properties of μ

Prop: $\# S_n^\lambda \varphi \asymp \lambda^{-n\delta}$

thm: $\mu(H) \asymp \lambda^{n\delta} \varphi(\lambda^n)^{2(\delta-\delta_\Pi)} (-2\log\varphi(\lambda^n)+1)^{a_\Pi}$

these are the "nice scaling properties" for the measure μ

combine to get what you want.

Since $\Theta_\lambda(\varphi) = \limsup S_n^\lambda \varphi$

Borel-Cantelli ① $\Rightarrow$ Khinchin theorem Ⓑ(1).

Conversely, Borel-Cantelli ② + quasi-independence lemma

$\Rightarrow \mu(\Theta_\lambda(\varphi)) > 0$.

To prove $\mu(\Theta_\lambda \varphi) = 1$, by ergodicity of μ wrt Γ, it suffices to show $\Theta_\lambda \varphi$ is invariant a.e.:

Lemma 5 (Stratmann) $\forall g \in \Gamma$,

$$\mu(g\,\Theta_\lambda \varphi \,\Delta\, \Theta_\lambda \varphi) = 0$$

i.e. $\mu(g\,\Theta_\lambda \varphi) = \mu(\Theta_\lambda \varphi)$.

(Ergodicity in this setting is due to Matsuzaki-Yabuki-Jaerisch)

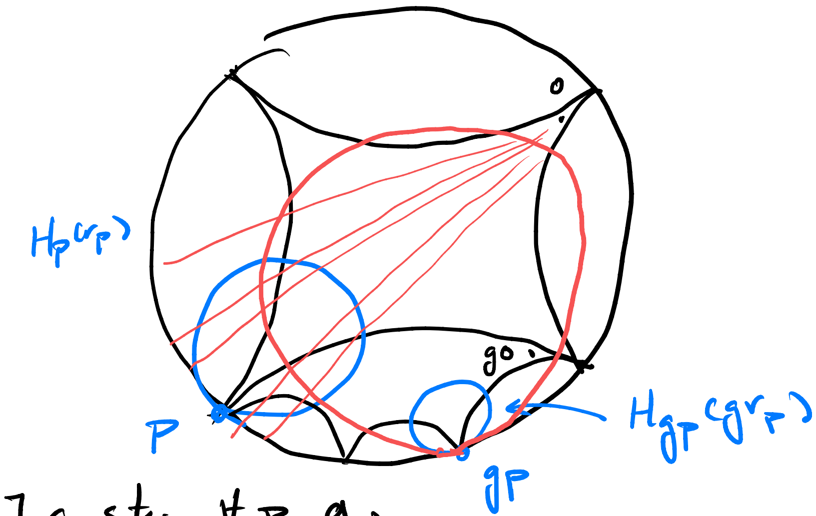
idea of proof:

It suffices to show $\forall g \in \Gamma$

$$\mu(g\,\Theta_\lambda \varphi) \leq \mu(\Theta_\lambda \varphi).$$

claim 1: $\forall g \in \Gamma$, $\exists\, c$ s.t. $\forall p \in \mathcal{P}$,

$$g\,H_p(r_p \varphi(r_p)) \subseteq H_p(c r_p \varphi(c r_p)).$$



$\exists\, c$ s.t. $\forall p, g,$

$$g H_p(r_p) \subseteq H_{gp}(c r_{gp}) \quad \text{by quasi-inv.}$$

arrange for: c' depending on $d(o, go)$ s.t.

$$g H_p(r_p) \subseteq H_p(c' r_p).$$

Now, we hope this implies $\forall c''$,

$$g H_p(c'' r_p) \subseteq H_p(c' c'' r_p) \quad \left(\begin{array}{l}\text{possibly incr } c'\\ \text{indep of } p\end{array}\right)$$

so letting $c'' = \varphi(r_p)$, we are done.

ran out of time to check this

Recall: $\varphi : \mathbb{R}^+ \to [0,1]$ Khinchin function if φ incr and $\exists\, b_1 < 1,\ b_2 > 0$ such that

$$\varphi(b_1 x) \geq b_2 \varphi(x)$$

$\forall x \in \mathbb{R}^+$ (important only for small x)

Note: constants are flexible

Let

$$\Theta_\lambda^c(\varphi) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \bigcup_{\lambda^{m+1} \leq r_p \leq \lambda^m} H_p(c\, r_p\, \varphi(r_p))$$

Claim 2:

$$\mu(\Theta_\lambda^c(\varphi)) = \mu(\Theta_\lambda(\varphi)).$$

Note: Lemma 5, hence Khinchin-type thm, follows.

it suffices to show

$$\mu\Big(\overbrace{\bigcup_{\lambda^{n+1}\le r_p\le\lambda^n} \mathcal{H}_p(c r_p, \varphi(r_p))}^{=E} \setminus \underbrace{\bigcup_{m\ge n}\bigcup_{\lambda^{m+1}\le r_p\le\lambda^m} \mathcal{H}_p(c r_p, \varphi(r_p))}_{=E_c}\Big) = 0$$

so assume >0 by contradiction.

Then $\exists$ density point $x \in E\setminus E_c$,
so $\exists$ seq $p_k \in \mathcal{P}$ w/ $r_{p_k} \to \infty$ s.t.
$x \in \underbrace{\mathcal{H}_{p_k}(c r_{p_k}, \varphi r_{p_k})}_{= H_k}$ $\forall k$ and decr. to x ("decr. metric balls abt x")

Then.

$$\frac{\mu(H_k \cap E\setminus E_c)}{\mu(H_k)} \overset{*}{\le} \frac{\mu(H_k) - \mu(E_c)}{\mu(H_k)}$$

$$\le 1 - \frac{\mu(E_c)}{\mu(H_k)} \Big\} \text{comparable by fine scaling properties} < 1 \quad \times$$

* still don't understand this one

Recall Lebesgue Density Thm: for a.e. $x \in A$,

$$\lim_{\varepsilon\to 0} \frac{\mu(A \cap B_\varepsilon(x))}{\mu(B_\varepsilon(x))} = 1.$$

x is a density pt if $=1$. See if $\mu(A)>0$ the $\exists$ density pt in A.