The Bishop-Jones Theorem

Day 1

Defn: G <u>Kleinian</u> iff $G < PSL(2,\mathbb{C}) \cong Isom(\mathbb{H}^3)$

and G discrete.

G <u>non-elementary</u> iff G has no fixed pts in $\partial\mathbb{H}^3 = S^2$

non-e.g.: $G = \langle z \mapsto \lambda z \rangle$, G parabolic

Defn: $\Lambda_G = \overline{G.0} \setminus G.0$ <u>limit set</u> of G.

Fact: G non-elem. iff $|\Lambda_G| > 3$

iff Λ_G uncountable

Defn: <u>conical limit set</u> Λ^c_G is all $x \in S^2$

s.t. $\exists$ cone $\mathcal{C}(x) \subseteq \mathbb{H}^3$ and $\{g_n\} \subseteq G$

s.t. $g_n 0 \in \mathcal{C}(x)$ and $g_n 0 \to x$.

Fact: G discrete $\Rightarrow$ G <u>properly discts</u>: $\forall K \subseteq \mathbb{H}^3$ cmpt,

$|\{g \in G : gK \cap K \neq \emptyset\}| < \infty$.

<u>exercise</u>: $g^+ = h^+$, g parabolic & h hyp $\Rightarrow$ $\langle g, h \rangle$ not properly discts.

$\left\langle \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\rangle$ θ irrational not properly discts.

Defn: critical exponent δ_G of G is absicca of convergence for

$$P(s) := \sum_{g \in G} (1 - |g0|)^s$$

Poincaré series.

Defn: $f \asymp g$ iff $\exists\, c$ s.t. $\frac{g}{c} \leq f \leq cg$.

exercise:

$d_{\mathbb{H}^3}(0, z) = \log \frac{|z|+1}{1-|z|}$ hence for $|z| > k$

$1 - |z| \asymp e^{-d_{\mathbb{H}^3}(0,z)}$

Thm: (Bishop-Jones)

$G < \mathrm{Isom}\, \mathbb{H}^3 \cong PSL(2, \mathbb{C})$ discrete, non-elem

then $\delta_G = \mathrm{H\,dim}\, \Lambda^c_G$

e.g. picture where Λ^c_G has nontrivial Hausdorff dimension.

Recall

$$H_\alpha^\delta(E) = \inf \left\{ \sum_{j=1}^{\infty} r_j^\alpha : E \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j),\ r_j < \delta \right\}$$

$$H_\alpha(E) = \lim_{\delta \to 0} H_\alpha^\delta(E)$$

$$\dim(E) = \inf \{ \alpha : H_\alpha(E) = 0 \}$$

Fact: $E \subseteq E' \Rightarrow \dim(E) \leq \dim(E')$

Pf: if $E' \subseteq \bigcup_j B(x_j, r_j)$ then so is E, so

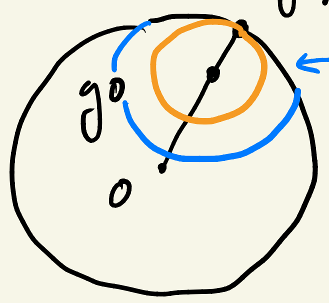
$$H_\alpha^\delta(E') \geq H_\alpha^\delta(E).$$

then take limits & infs. $\square$

Pf:

$\delta_G \geq \Lambda_G^c$ easy direction

$g0/|g0| = \xi_g$ $\quad B_g = B\left(\frac{g0}{|g0|}, M(1-|g0|)\right)$



$E_M = \{\xi \in \partial \mathbb{H}^n \mid \xi \in \text{infntly many } B_g\}$

see that $\forall\, \varepsilon > 0$,

$$\sum_{g \in G} \operatorname{diam}(B_g)^{\delta+\varepsilon} \simeq \sum_{g \in G} (1-|g0|)^{(\delta+\varepsilon)} < \infty .$$

$$E_M \subseteq \bigcup_{\substack{g \in G \\ d_{\mathbb{H}^3}(0, g0) > N}} B_g \quad , \text{ so}$$

$$\sum_{\substack{g \in \Gamma \\ d_{\mathbb{H}^3}(0, g0) > N}} \operatorname{diam}(B_g)^{\delta_G+\varepsilon} \to 0 \quad \text{as } N \to \infty.$$

$\forall\, \delta > 0$, $\exists\, N$ large so that $d(0, g0) > N \Rightarrow$ $\operatorname{diam} B_g < \delta$, so for δ_G we have shown

$H^{\delta}_{\delta_G+\varepsilon}(E_M) = 0$ hence $H_{\delta_G+\varepsilon}(E_M) = 0$.

Then $\dim E_M \leq \delta_G + \varepsilon$. exercise

Since $\boxed{\Lambda_G^c \subseteq E_M \text{ for } M \text{ suff large,}}$ and $\varepsilon > 0$ arbitrary,

we conclude

$$\dim \Lambda_G^c \leq \delta_G .$$

Day 2

Last time

Thm: (Bishop-Jones)

$G < \mathrm{Isom}\,\mathbb{H}^3 \cong PSL(2,\mathbb{C})$ discrete, non-elem

then
$$\delta_G = H\dim \Lambda^c_G$$

recall δ_G critical exponent of Poincaré series

$$P(s) := \sum_{g\in G} (1-|g0|)^s$$

Alt defn: $\Lambda^c_G = \{ \lim g_n 0 : \bigcup_{n\in\mathbb{N}} [g_n 0, g_{n+1} 0] \}$

is a quasi-geodesic ray ie the image of a

map $\varphi: \mathbb{R}^+ \to \mathbb{H}^3$ such that

$$\frac{1}{A}|t-s| - B \leq d_{\mathbb{H}^3}(\varphi(t),\varphi(s)) \leq A|t-s| + B.$$

exercise: verify this alternate definition

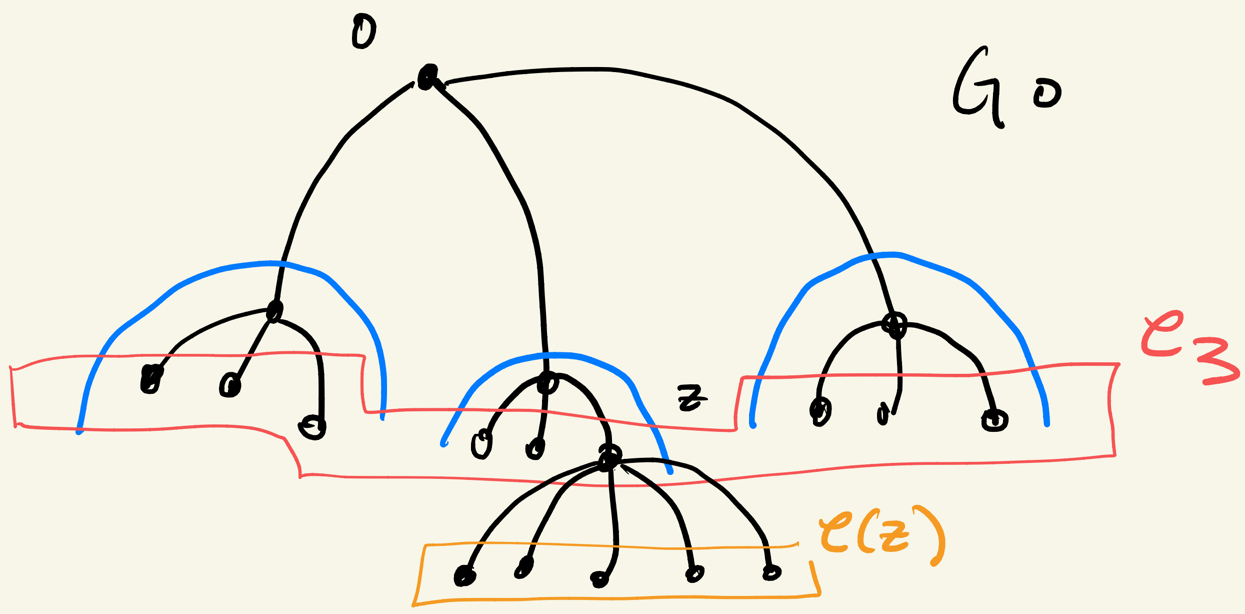
exercise: $\Lambda^c_G \subseteq E_M$ for M suff. large

using forthcoming content.

$\delta_G \le \dim \Lambda_G^c$ hard direction

Let $\varepsilon > 0$. Let $\delta = \delta_G$.

tree of orbit points



$$\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$$

$$E = \partial \mathcal{C}$$

goal: $E \subseteq \Lambda_\Gamma^c$ and $\dim E \geqslant \delta_\Gamma - \varepsilon$

Strategy construct $\mathcal{C}$ s.t.

Prop 1) $w \in \mathcal{C}(z) \Rightarrow w \in B(z, N(1-|z|)$ excl!

Prop 2) $w \in \mathcal{C}(z) \Rightarrow C_0^{-1} \le \dfrac{(1-|z|)}{(1-|w|)} \le \frac{1}{2}$

Prop 3) $w_1 \neq w_2 \in \mathcal{C}(z) \Rightarrow B(w_1, 2N(1-|w_1|)) \cap B(w_2, 2N(1-|w_2|)) = \phi$

Prop 4) $\displaystyle\sum_{w \in \mathcal{C}(z)} (1-|w|)^{(\delta - 2\varepsilon)} \geqslant C_0^2 (1-|z|)^{(\delta - 2\varepsilon)}$

1. construction

2. verify the properties

3. justify $E \subseteq \Lambda_\Gamma^c$.

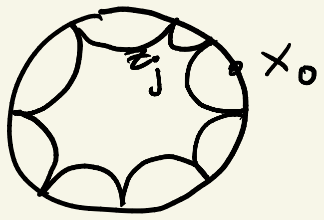
4. prove $\mathrm{Hdim}\, E \geq \delta - 2\varepsilon$

construction

$\{z_n\} = G.0$

clm: $\exists\ x_0 \in S^2$ s.t. (*) $\displaystyle\sum_{\substack{j\in\mathbb{N} \\ |z_j - x_0|<r}} (1-|z_j|)^{\delta-\varepsilon} = \infty \quad \forall\ r>0,\ \varepsilon>0.$

pf: δ is chosen so that $\displaystyle\sum_{n\in\mathbb{N}} (1-|z_n|)^{\delta-\varepsilon} = \infty.$



z_n accumulate on S^2. cover S^2 with finitely many such nhds radius $\frac{1}{k}$. Then

divergence $\Rightarrow\ \exists\ x_k$ s.t.

(**) $\displaystyle\sum_{\substack{j\in\mathbb{N} \\ |z_j - x_k|<\frac{1}{k}}} (1-|z_j|)^{\delta-\varepsilon} = \infty$ (since G is discrete, hence acts properly discontinuously)

Now let x_0 be an accumulation point of x_k.

$\forall\ r,\ \exists\ k$ large, $|x_k - x_0| < \frac{r}{2}$ and $\frac{1}{k} < \frac{r}{2}$ so

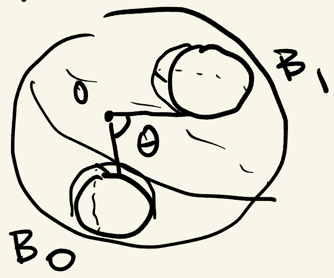
so $|z_j - x_k| < \frac{1}{k} \Rightarrow |z_j - x_0| < r.$

Then (**) $= \infty \Rightarrow$ (*) $= \infty$. ☐

G non-elem $\Rightarrow\ \exists\ g\in G$ s.t. $x_1 = g x_0 \neq x_0$.

Let $B_i = B(x_i, r)$ $i=0,1$ where r is small enough that $\forall$ geod. rays γ_0, γ_1 w/ $\gamma_i^+ \in B_i$, $\measuredangle_0(\gamma_0,\gamma_1) > \theta$ for some θ

(This is just Euclidean geometry now)



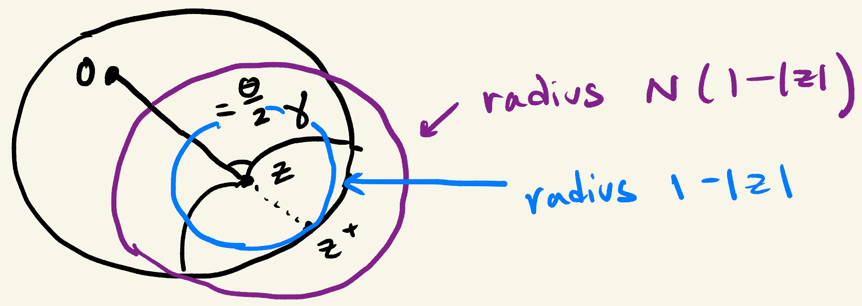
exercise

$\displaystyle\sum_{z_j\in B_i} (1-|z_j|)^{\delta-\varepsilon} = \infty$ for $i=0,1$

Property A

$\exists N$ s.t. $z \neq 0 \Rightarrow \forall$ geod. ray γ at z s.t. $\measuredangle_z(\gamma, [0,z]) > \frac{\theta}{2}$,

$\gamma \subseteq B(z, N(1-|z|))$.



proof can be rephrased in terms of Gromov product (Paulin)

pf: hyperbolic geometry exercise?

Property B $\quad \exists K \in [1,\infty)$ s.t. if $\measuredangle_z([z,w],[0,z]) > \frac{\theta}{2}$,

$d(z,w) \geq L \Rightarrow$

$$1 \leq \frac{(1-|w|)e^L}{1-|z|} \leq K$$

if $L > \log 2K$, then $2K < e^L$ so

$$1 \leq \frac{1-|w|}{1-|z|} e^L \leq K \leq \frac{e^L}{2} \Rightarrow$$

$$e^{-L} \leq \frac{1-|w|}{1-|z|} \leq \frac{1}{2}.$$

Let $C_0 = e^L$.

Let $A_n = \{ z \in \mathbb{H}^3 : 2^{-(n+1)} \leq 1-|z| \leq 2^{-n} \}$.

clm: for $i = 0, 1$

$$\varlimsup_{n\to\infty} \sum_{z_j \in B_i \cap A_n} (1-|z_j|)^{\delta - 2\varepsilon} = \infty \quad \oplus$$

pf: Assume false. Then $\exists M$ s.t. $\forall n$,

$$\sum_{z_j \in B_i \cap A_n} (1-|z_j|)^{\delta-2\varepsilon} \leq M < \infty$$

Then

$$\sum_{z_j \in B_i} (1-|z_j|)^{\delta-\varepsilon} = \sum_{n\in\mathbb{N}} \sum_{z_j \in B_i \cap A_n} (1-|z_j|)^{\delta-\varepsilon} \qquad 1-|z_j| \le 2^{-n}$$

$$= \sum_{n\in\mathbb{N}} \sum_{z_j \in B_i \cap A_n} (1-|z_j|)^{\delta-2\varepsilon} (1-|z_j|)^{\varepsilon}$$

$$< \sum_{n\in\mathbb{N}} \sum_{z_j \in B_i \cap A_n} (1-|z_j|)^{\delta-2\varepsilon} \, 2^{-n\varepsilon}$$

$$\le \sum_{n\in\mathbb{N}} \left(\frac{1}{2^{\varepsilon}}\right)^n M \qquad \varepsilon>0 \text{ so } 2^{\varepsilon} > 2^0 = 1$$

$$< \infty$$

Clm: $\oplus \iff \overline{\lim_{n\to\infty}} \dfrac{\# z_j \in B_i \cap A_n}{2^{n(\delta-2\varepsilon)}} = \infty$ $\boxed{++}$

exercise prove this

Since G acts properly discretely, $\forall A<\infty$, $\exists B$ s.t.

$\# G.0 \cap \overline{B_{\mathbb{H}^3}(0,A)} < B$.

Then write $G.0$ as a union of B segs $(z_n^1),\dots,(z_n^B)$

s.t. $\forall n \neq m$, $\forall i$, $d(z_n^i, z_m^i) > A$.

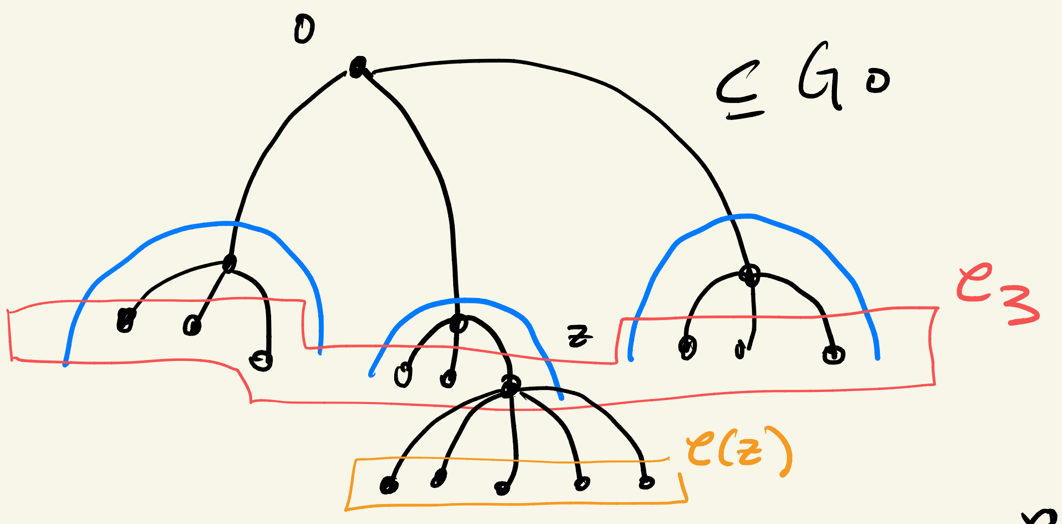
Property C Given N, $\exists A$ s.t. $\forall w_1, w_2 \in A_n$ w/ $d_{\mathbb{H}^2}(w_1,w_2) \ge A$,

$$B(w_1, 3N(1-|w_1|)) \cap B(w_2, 3N(1-|w_2|)) = \emptyset$$

Day 3

Last time

construction



$$e = \bigcup_{n=0}^{\infty} e_n$$

$\{z_n\} = G.0$

$A_n = \{ z \in \mathbb{H}^3 : 2^{-(n+1)} \leq 1 - |z| \leq 2^{-n} \}.$

$\exists$ $x_0, x_1 = g x_0 \neq x_0$ and $B_i = B(x_i, r)$ s.t.

$$\varlimsup_{n\to\infty} \sum_{z_j \in B_i \cap A_n} (1 - |z_j|)^{\delta - 2\varepsilon} = \infty$$

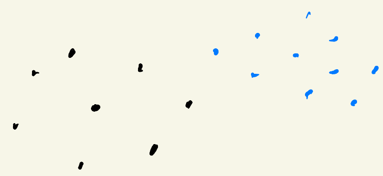
Hence

$$\varlimsup_{n\to\infty} \frac{\# z_j \in B_i \cap A_n}{2^{n(\delta - 2\varepsilon)}} = \infty$$

$\# z_j \in B_i \cap A_n \xrightarrow[n\to\infty]{} \infty$

Continuing

Since G acts properly discretely, $\forall A < \infty$, $\exists B$ s.t.



$\# G.0 \cap \overline{B_{\mathbb{H}^3}(0,A)} < B.$

Then write $G.0$ as a union of B segs $(z_n^1), \dots, (z_n^B)$

s.t. $\forall n \neq m$, $\forall i$, $d(z_n^i, z_m^i) > A$.

Property C Given N, $\exists A$ s.t. $\forall w_1, w_2 \in A_n$ w/ $d_{\mathbb{H}^2}(w_1, w_2) \geq A$,

$$B(w_1, 3N(1 - |w_1|)) \cap B(w_2, 3N(1 - |w_2|)) = \emptyset$$

exercise: $\frac{1}{2} \le \frac{1-|z|}{1-|w|} \le 2 \Rightarrow d_{\mathbb{H}^3}(0,z) \approx d(0,w)$ (as long as $|w_i| > k$, $\approx$ dep on k)

solution: $d_{\mathbb{H}^3}(0, z) = \log \frac{|z|+1}{1-|z|}$

$$= \log(|z|+1) - \log(1-|z|)$$

$|z|, |w| > k$ then

$$\log\left(\frac{k+1}{2}\right) \le \log\left(|z|+1/_{|w|+1}\right) \le \log\left(2/_{k+1}\right)$$

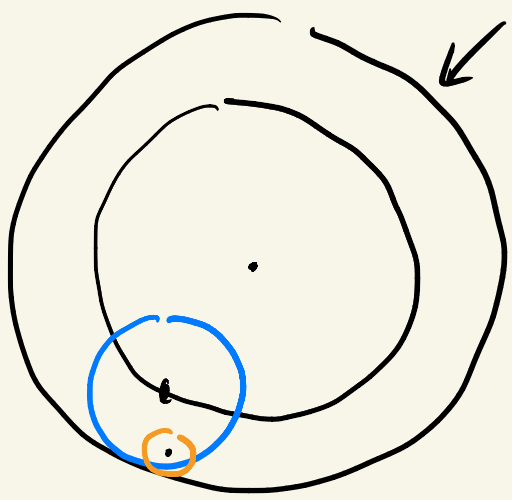
Then

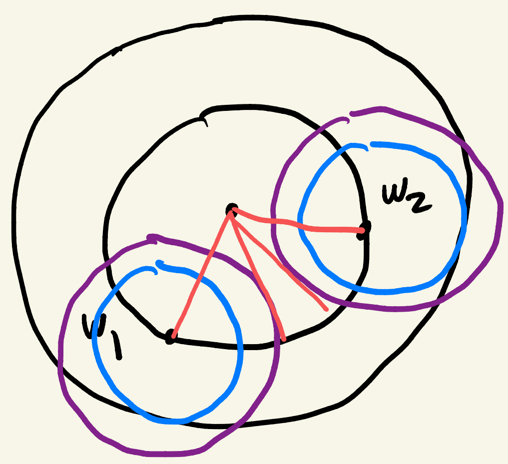
$$d_{\mathbb{H}^3}(0,w) - c \le d_{\mathbb{H}^3}(0,z) \le d_{\mathbb{H}^3}(0,w) + c$$

where $c = \log\left(\frac{2}{k+1}\right) + \log 2$. $\square$

Pf: of Prop C. $w_i \in A_n \Rightarrow d_{\mathbb{H}^3}(0, w_1) \approx d_{\mathbb{H}^3}(0, w_2)$ by exercise above.

Idea is to then make $d_{\mathbb{H}^2}(w_1, w_2) \geq A$ large, forcing $\measuredangle_0([0,w_1],[0,w_2]) > 3\theta$ or something (note requires them to be same annulus around 0), which makes the balls disjoint since they form a definite angle.





Claim for $i=0,1$ $\exists$ $V_i \subseteq \{z_n\} = G.0$ assoc. to x_i

s.t. $w_1 \neq w_2 \in V_i \Rightarrow$

$$B(w_1, 3N(1-|w_1|)) \cap B(w_2, 3N(1-|w_2|)) = \phi$$

and

$$\sum_{z_j \in V_i} (1-|z_j|)^{\delta-2\varepsilon} \geq e^{-(\delta-2\varepsilon)L_i} \# V_i \quad \text{needed?}$$

$$\geq c_0^4$$

Pf: Fix A as in Property C.

$G.0 = \bigcup_{i=1}^{B}$ segs pairwise dist $\geq A$

so disjtness follows Prop C by taking V_i to be a subset of any such sequence $\{z_n^k\}$

pick $V_i = \{z_n^k\} \cap B_i \cap A_{n_i}$, for some n_i tbd. $i=0,1$

See that $z_n^1, \dots, z_n^B$ bndd distance

$$\# z_j \in B_i \cap A_{n_i} \leq B \# z_j \in \{z_n^k\} \cap B_i \cap A_{n_i}$$

maybe some wiggling needed, not being optimally careful here

Hence

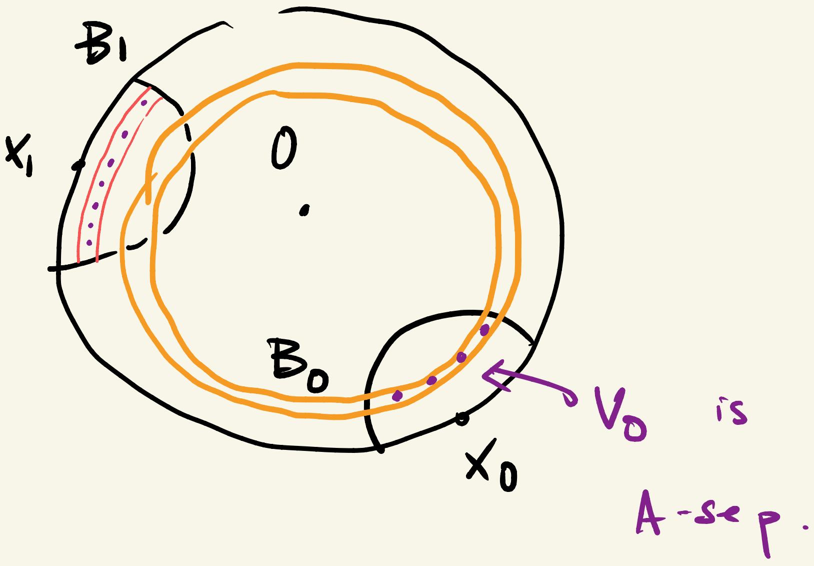
$$\oplus \Rightarrow \overline{\lim_{n\to\infty}} \sum_{z_j \in \{z_n^k\} \cap B_i \cap A_n} (1-|z_j|)^{\delta-2\varepsilon} = \infty$$

hence $\exists n_i$ s.t.

$$\sum_{z_j \in V_i} (1-|z_j|)^{\delta-2\varepsilon} \geq c_0^4$$

b/c $\oplus\!\!\oplus \Rightarrow \# V_i \to \infty$.

Note that by taking n_i large, we can ensure $z \in V_i \Rightarrow z \in A_{n_i} \Rightarrow$ $1-|z| \le 2^{-n_i}$ small $\Rightarrow d_{\mathbb{H}^3}(0,z) \geq 2\log C_0$. $(+*)$



Time to define $\mathcal{C} = \cup \mathcal{C}_n$

by induction

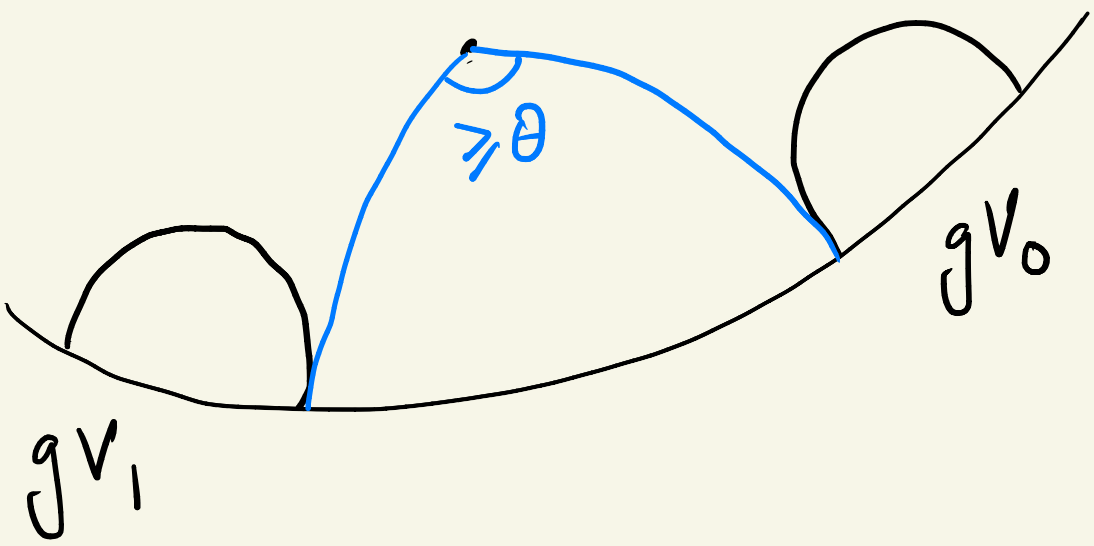
$\mathcal{C}_0 = \{0\}$, $\mathcal{C}_1 = V_0$

If $z = g0 \in \mathcal{C}_n$, $\measuredangle_z([z, gV_0], [z, gV_1]) \geq \theta$

by construction of V_0, V_1 and g conformal.

Then $\exists\, i \in \{0,1\}$ s.t.

$\measuredangle_z([0,z], [z, gV_i]) \geq \frac{\theta}{2}$.



Also note:

$d_{\mathbb{H}^3}(z, gV_i) \geq 2\log C_0$.

Set $\mathcal{C}(z) := gV_i$,

and $\mathcal{C}_{n+1} = \bigcup_{z \in \mathcal{C}_n} \mathcal{C}(z)$.

<u>Remark</u>: the construction is not G-invariant.

Now, we verify Properties (1)–(4).

Prop 1) $w \in \mathcal{C}(z) \Rightarrow w \in B(z, N(1-|z|)$

Prop 2) $w \in \mathcal{C}(z) \Rightarrow C_0^{-1} \le \frac{(1-|z|)}{(1-|w|)} \le \frac{1}{2}$

Pf: By construction, $\measuredangle_z([0,z],[z,w]) \geq \frac{\theta}{2}$

where θ is given in Property A & B.

1) now follows prop A

2) follows Prop B and that $d(0,z) \geq 2 \log C_0$

as in (+*)

Prop 3) $w_1 \neq w_2 \in \mathcal{C}(z) \Rightarrow B(w_1, 2N(1-|w_1|)) \cap B(w_2, 2N(1-|w_2|)) = \phi$

Pf: $w_1, w_2 \in A_{n_i}$ some $i \Rightarrow d_{\mathbb{H}^3}(w_1,w_2) \geq A$.

Conclusion follows Property C. □

Prop 4) $\sum_{w \in \mathcal{C}(z)} (1-|w|)^{(\delta-2\varepsilon)} \geq C_0^2 (1-|z|)^{(\delta-2\varepsilon)}$

Pf.

$$\sum_{w \in \mathcal{C}(z)} (1-|w|)^{\delta-2\varepsilon}$$

$$\geq \sum_{w \in \mathcal{C}(z)} 2^{\delta-2\varepsilon} (1-|z|)^{\delta-2\varepsilon} \quad \text{(property 2)}$$

$$\geq 2^{\delta-2\varepsilon} (1-|z|)^{\delta-2\varepsilon} \# \mathcal{C}(z)$$

$$= (1-|z|)^{\delta-2\varepsilon} \# V_i \, 2^{\delta-2\varepsilon}$$

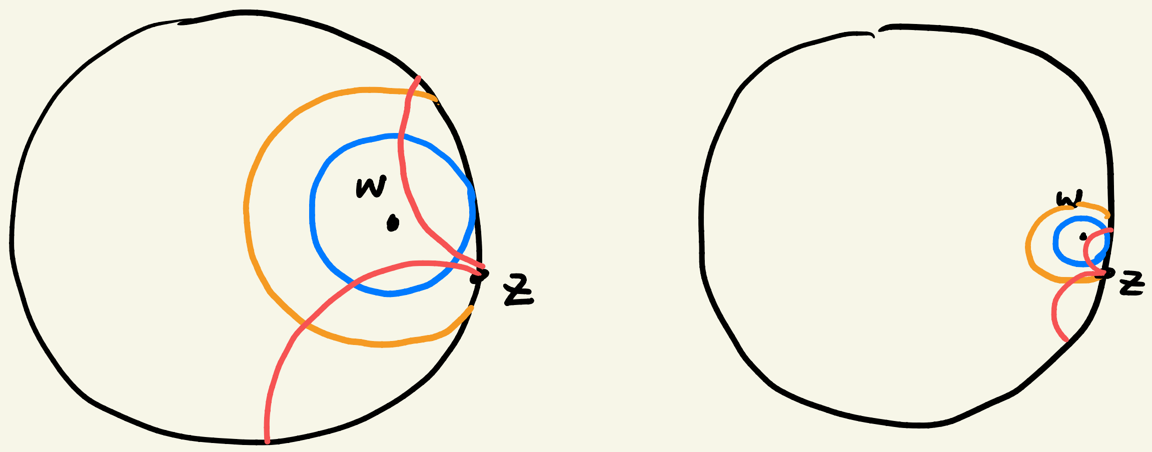
and $\# V_i \geq C_0^2 / 2^{\delta-2\varepsilon}$ for i large. □

Recall $E = \partial \mathcal{C}$

<u>claim</u>: $E \subseteq \Lambda_G^c$

<u>Pf</u>: <u>Property D</u> If $z \in S^2$, $z \in B(w, 2N(1-|w|))$ then $w \in \mathrm{Cone}(z)$ of angle only dep. on N.

start here



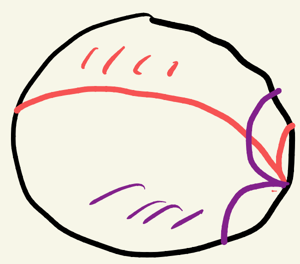
If $z \in E$, then $\exists$ ∞-ly many $z_n \in \mathcal{C}$ s.t.

$$z \in B(z_n, 2N(1-|z_n|))$$

Since $\{B(w,r) : w \in \mathbb{H}^3\}$ generates the topology on $\overline{\mathbb{H}^3}$ for any fixed r.

Then Prop. D $\Rightarrow$ $z_n \in \mathrm{Cone}_n(z)$ $\forall n$, and each cone_n has the same angle.

Goal: $\mathrm{cone}_n = \mathrm{cone}_m$ for n, m large.

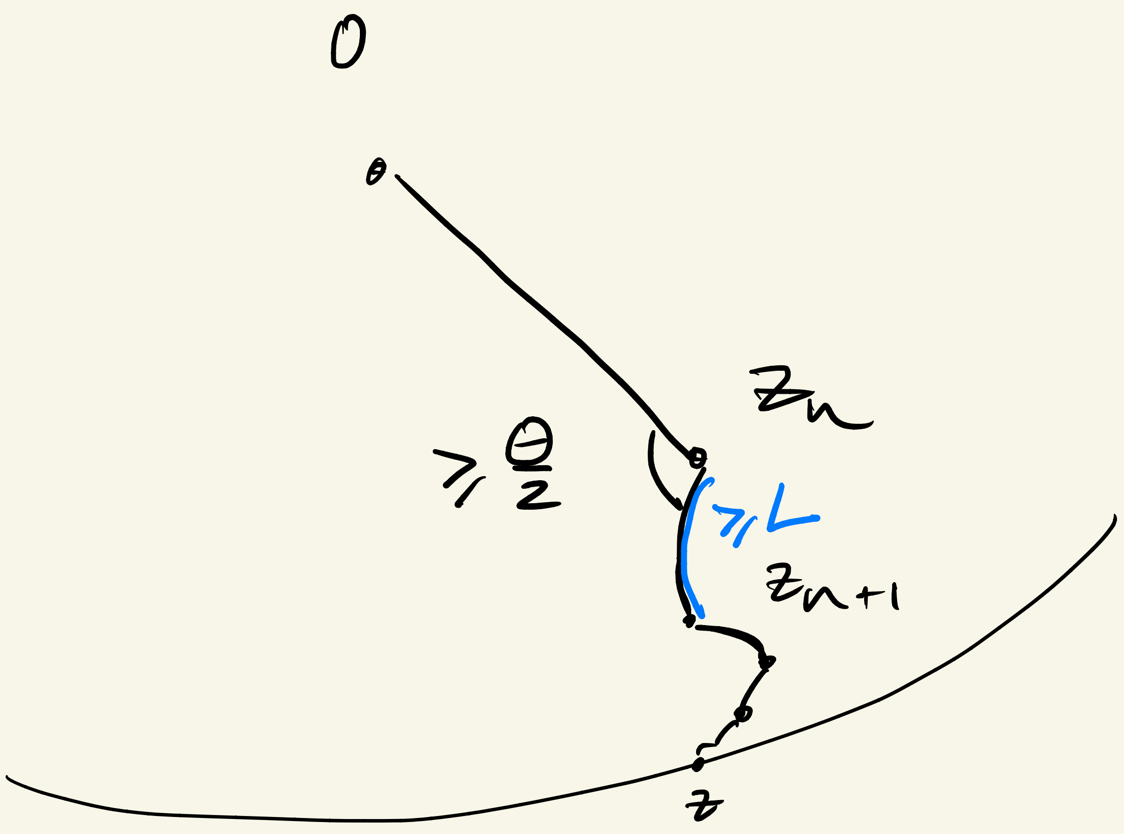


alternate proof

If $z_{n+1} \in \mathcal{C}(z_n)$, then we saw

(★) $\measuredangle_z([0, z_n], [z_n, z_{n+1}]) \geq \frac{\theta}{2}$.

and $d(z_n, z_{n+1}) \geq 2 \log c_0$



Clm: $\bigcup [z_n, z_{n+1}]$ is a quasi-geodesic, hence by the exercise, $z \in \Lambda_G^c$

Local to global principle:

$(*) \Rightarrow \quad d_{\mathbb{H}^3}(z_n, z_{n+m}) \geq \sum_{i=n}^{n+m} d(z_i, z_{i+1}) - (m-1)c$

hence $\quad \geq mL - (m-1)c.$

upper bound:

$$d_{\mathbb{H}^3}(z_n, z_{n+m}) \leq \sum_{i=n}^{n+m} d(z_i, z_{i+1}) \leq mL'$$

by property 2 and exercise.

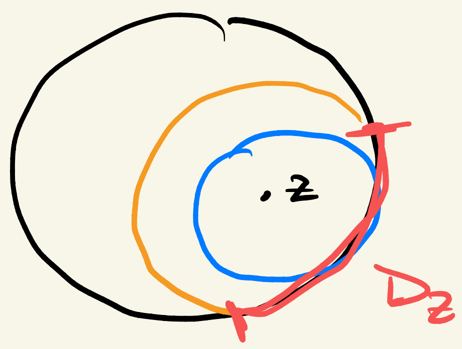
This is defn of quasi geod.

$\square$

Resume here

Now, the calculation with Hausdorff dimension.

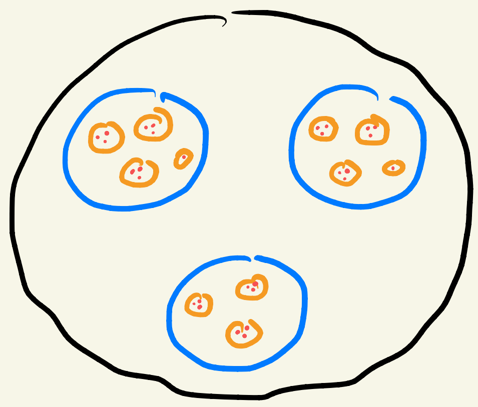
"This is a standard argument"



$D_z := B(z, 2N(1-|z|)) \cap S^2$

Let $E_n = \bigcup_{z \in \mathcal{C}_n} D_z$

Clm: $E_{n+1} \subseteq E_n$



In particular, $z \in \mathcal{C}(z') \Rightarrow D_z \subseteq D_{z'}$.

Cor: $E = \limsup_{n\to\infty} E_n = \bigcap_{n \in \mathbb{N}} E_n$.

Pf of clm:

Let $x \in D_z$ for some $z \in \mathcal{C}_{n+1}$. Then

$$|x - z| \leq 2N(1-|z|).$$

Let $z \in \mathcal{C}(z')$ for $z' \in \mathcal{C}_n$.

Then by Property 1, $z \in B(z', N(1-|z'|))$.

Then Δ-ineq $\Rightarrow$

$$\begin{aligned} |x - z'| &\leq |x-z| + |z-z'| \\ &\leq 2N(1-|z|) + N(1-|z'|) \\ &\leq 2N\left(\tfrac{1}{2}(1-|z'|)\right) + N(1-|z'|) \qquad \text{Prop 2*} \\ &= 2N(1-|z'|). \end{aligned}$$

Thus $x \in D_{z'}$. $\square$

Fact: If $\exists$ Borel μ on X s.t. $\mu(X) > 0$

and $\mu(B(x,r)) \leq r^s$ holds for some $s \geq 0$, $\forall x \in X$

then $\quad \mathrm{Hdim}(X) \geq s$.

Define prob μ on E by $\mu(E_0) = 1$

If $\quad z \in \mathcal{C}(z') \subseteq \mathcal{C}_n$,

$$\mu(D_z) = \frac{(1-|z|)^{\delta-2\varepsilon}}{\sum_{w \in \mathcal{C}(z')} (1-|w|)^{\delta-2\varepsilon}} \mu(D_{z'})$$

diam $D_z \simeq$ radius of ball

Clm: $\quad \mu(D_z) \leq C\,(1-|z|)^{\delta-2\varepsilon} \leq C' \operatorname{diam}(D_z)^{\delta-2\varepsilon}$

Pf: by induction on $\mathcal{C}_n$.

First make C large enough that the base case is true.

see that $c' = (2N)^{\delta-2\varepsilon} C$.

If true for z', then

$$\mu(D_z) = \frac{(1-|z|)^{\delta-2\varepsilon}}{\sum_{w \in \mathcal{C}(z')} (1-|w|)^{\delta-2\varepsilon}} \mu(D_{z'})$$

$$\leq \frac{(1-|z|)^{\delta-2\varepsilon}}{(1-|z'|)^{\delta-2\varepsilon}} \mu(D_{z'}) \qquad \text{property 4}$$

$$\leq C(1-|z|)^{\delta-2\varepsilon} \frac{\cancel{(1-|z'|)^{\delta-2\varepsilon}}}{\cancel{(1-|z'|)^{\delta-2\varepsilon}}} \qquad \text{by induction}$$

$$= C\,(1-|z|)^{\delta-2\varepsilon}.$$

$\square$

clm: $\mu(E) = 1$.

Pf: $E_{n+1} \subseteq E_n$ and μ is Borel $\Rightarrow$

$$\mu(E) = \mu(\cap E_n) = \lim \mu(E_n)$$

Inductively,

$$\mu(E_{n+1}) = \sum_{z \in \mathcal{C}_1} \mu(D_z)$$ pairwise disjt by Property 3

$$= \sum_{z' \in \mathcal{C}_n} \sum_{z \in \mathcal{C}(z')} \mu(D_z)$$

$$= \sum_{z' \in \mathcal{C}_n} \sum_{z \in \mathcal{C}(z')} \frac{(1-|z|)^{\delta - 2\varepsilon}}{\sum_{w \in \mathcal{C}(z')} (1-|w|)^{\delta - 2\varepsilon}} \mu(D_{z'})$$

$$= \sum_{z' \in \mathcal{C}_n} \mu(D_{z'}) \cancel{\frac{\sum_{z \in \mathcal{C}(z')} (1-|z|)^{\delta - 2\varepsilon}}{\sum_{w \in \mathcal{C}(z')} (1-|w|)^{\delta - 2\varepsilon}}}$$

$= \mu(E_n)$ pairwise disjt by Property 3

$= 1$ induction.

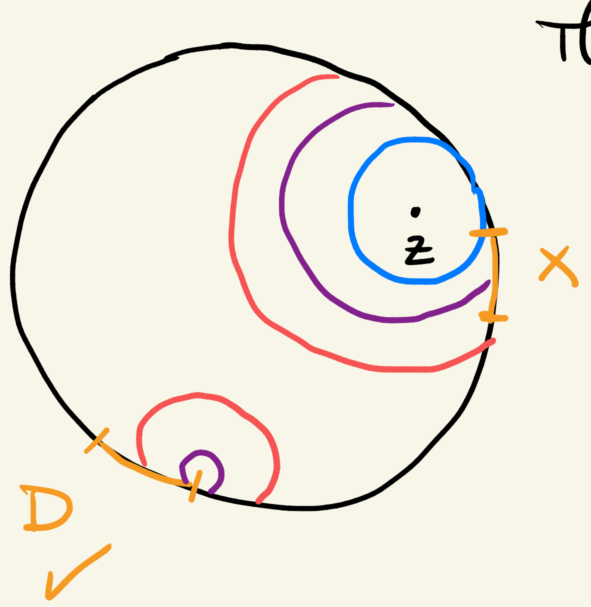
clm: $\forall\, D \subseteq S^2$ s.t. $D \cap E \neq \emptyset$,

$$\mu(D) \lesssim \operatorname{diam} D^{\delta - 2\varepsilon}$$

The result follows the fact since $\mu(E) = 1 > 0$,

Pf of clm:

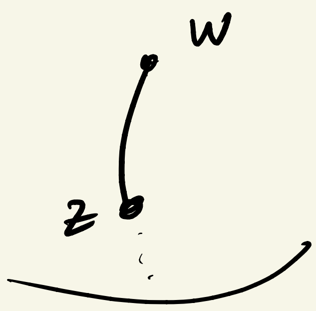
Let $D_0 = D_z$ for $z \in \mathcal{C}(n)$ where n is the smallest value so that $\frac{1}{2}D_z \cap D \neq \emptyset$ but $D \not\subseteq D_z$



This is possible since radii shrinking and $D \cap E \neq \emptyset$ so ∞-ly many $D_z \cap E \neq \emptyset$.

Let $D_1 = D_w$ where $z \in \mathcal{C}(w)$. So $D_0 \subseteq D_1$ as we proved before.

By minimality of n, $D \subseteq D_1$.



Then

$$\mu(D) \leq \mu(D_1) \leq C \operatorname{diam}(D_1)^{\delta - 2\varepsilon}.$$

$1 - |z| \leq \frac{1}{2}(1 - |w|)$

Note that

$$\operatorname{diam} D_1 = 2N(1 - |w|) \leq 2N(C_0(1 - |z|)) \quad \text{by prop. 2}$$

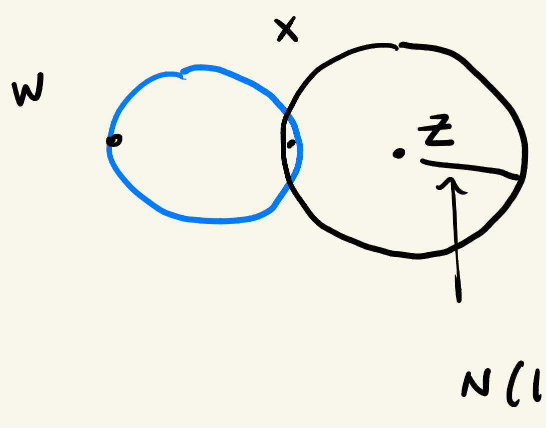
$$\leq C'' \operatorname{diam} \tfrac{1}{2} D_0$$

$\operatorname{diam} D_z \asymp \text{rad}$

$$\mu(D) \le C(2NC_0)^{\delta-2\varepsilon}\left(\operatorname{diam}\tfrac{1}{2}D_0\right)^{\delta-2\varepsilon}$$

D_0 disjt from other balls of same generation by property 3.

clm: $\operatorname{diam}\tfrac{1}{2}D_0 \le c''' \operatorname{diam} D$.



Pf: if $\operatorname{diam} D < \frac{1}{c'''}\operatorname{diam}\tfrac{1}{2}D_0$

$$\le \frac{1}{c'''}c'''N(1-|z|)$$

$$= N(1-|z|),$$

then $w \in D \Rightarrow$

$$d(w,z) \le d(w,x) + d(x,z)$$
$$\le N(1-|z|) + N(1-|z|)$$
$$\le 2N(1-|z|)$$

$$\Rightarrow w \in B(z, 2N(1-|z|)) \cap S^2 = D_z.$$

$\square$

Thus,

$$\mu(D) \lesssim \left(\operatorname{diam}\tfrac{1}{2}D_0\right)^{\delta-2\varepsilon}$$
$$\lesssim (\operatorname{diam} D)^{\delta-2\varepsilon}$$

and the proof is complete. $\square$

pf of Frostman's Lemma

$$\mathcal{H}^d(X) = \inf \left\{ \sum_{i\in\mathbb{N}} \operatorname{diam} U_i^{\,d} : \bigcup_{i\in\mathbb{N}} U_i \supseteq X \right\}$$

$$\dim_H(X) = \inf \{ d : \mathcal{H}^d(X) > 0 \}$$

$=0$

Then $s \leq \dim_H(X)$ iff $\mathcal{H}^s(X) > 0$.

Take a cover U_i

$$\sum_i (\operatorname{diam} U_i)^s = \sum_i \operatorname{rad}(B(x_i, \operatorname{diam} U_i))^s$$

$$\geq \sum \mu(B(x_i, r_i))$$

$$\geq \mu(X) > 0. \qquad \square$$