A GLOBAL SHADOW LEMMA AND LOGARITHM LAW FOR GEOMETRICALLY FINITE HILBERT GEOMETRIES

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Abstract. For geometrically finite group actions on hyperbolic metric spaces and under certain assumptions on the growth of parabolic subgroups, we prove a global shadow lemma for Patterson–Sullivan measures, as well as a Dirichlet-type theorem and a logarithm law for excursion of geodesics into cusps. We then apply these results to geometrically finite quotients of strictly convex Hilbert geometries with C^1 boundary.

1. Introduction

In this work, we prove a global version of the *shadow lemma* ([Sul79, Sul84, SV95]) for the Patterson–Sullivan measures associated to geometrically finite, strictly convex real projective manifolds. We then apply it to obtain a *logarithm law*, as in [Sul82], which provides asymptotics for the maximal cusp excursion for generic geodesics and relates it to the dimension of the limit set. Our results follow from more general statements we will prove in the context of hyperbolic metric spaces which satisfy certain growth conditions for the parabolic subgroups.

A convex real projective structure is given by a properly convex domain Ω in real projective space $\mathbb{R}P^n$, with an action by a discrete group Γ of projective transformations preserving Ω. The quotient manifold $M = \Omega/\Gamma$ is called a *convex projective manifold*, and inherits a natural metric d_Ω called the *Hilbert metric*. If Ω is strictly convex, geodesics for the Hilbert metric are simply straight lines. The moduli space of these geometries is frequently nontrivial and includes the example of hyperbolic structures of constant negative curvature.

A Hilbert geometry (Ω, d_Ω) is in general only Finsler, meaning the metric comes from a norm, but this norm does not necessarily come from an inner product. Once Ω is preserved by a noncompact group of projective transformations, the Hilbert geometry (Ω, d_Ω) is Riemannian if and only if Ω is an ellipsoid ([SM02], [Cra14, Theorem 2.2]). Moreover, aside from the special case of the ellipsoid, these Hilbert geometries are not $\mathrm{CAT}(k)$ for any k [Mar14]. Nonetheless, as Marquis states in [Mar14], we may think of Hilbert geometries as having "damaged nonpositive curvature." In particular, a strictly convex Hilbert geometry with a large isometry group has many properties resembling negative curvature.

Hyperbolic manifolds are equipped with a natural boundary of their universal cover which carries several interesting quasi-invariant measures; in particular, the *Patterson–Sullivan* measure, obtained by taking limits of Dirac measures supported on the group orbit ([Pat76, Sul79, Sul84]). In that context, Sullivan's *shadow lemma* establishes the scaling properties of such measures near boundary points. These properties turn out to depend subtly on the location of the parabolic points, and are related to the fine structure of the limit set.

In this paper we are going to study, more specifically, geometrically finite convex projective manifolds, and extend this result to them. In this context and in various degrees of generality,

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an analogue of the Patterson–Sullivan measure has been constructed on the boundary of Ω in projective space (see [Ben04, Cra11, CM14b, Zhu20, Bra20, Bla21, BZ23]).

Let us now assume that Γ is a geometrically finite group of isometries of a strictly convex domain Ω with C^1 boundary and that the convex hull C_Γ of the limit set Λ_Γ is hyperbolic in the sense of Gromov.

This assumption applies to a large family of examples; for instance, if Ω is strictly convex with C^1 boundary, any properly discontinuous action with cofinite volume, and more generally any geometrically finite action for which all parabolic stabilizers have maximal rank, will have a Gromov hyperbolic convex hull [CLT15, CM14a]. The moduli space of finite volume convex real projective structures on a surface of genus g with p punctures has real dimension $16g - 16 + 8p$ [Mar10]. In these settings, the parabolic stabilizers are conjugate into $SO(n, 1)$ [CLT15, CM14a]. There are moreover examples where the parabolic stabilizers are not conjugate into $SO(n, 1)$ [CM14a, Proposition 10.7] and yet the convex hull is still hyperbolic [DGK21, Zim21], so our results apply. We expand on this discussion in Subsection 7.2.

To state the theorem, we now introduce a few definitions. For a basepoint o and a boundary point ξ, let ξ_t be the point on a geodesic ray from o to ξ at distance t from o.

Definition 1.1. Define the *shadow* $V(o, \xi, t)$ from a point $o \in \Omega$ to a boundary point ξ of depth $t \geq 0$ to be the set of all boundary points $\eta \in \partial\Omega$ such that the Gromov product $\langle \xi, \eta \rangle_o$ is at least t (see Subsection 2.1).

Essentially, a boundary point η belongs to the shadow $V(o, \xi, t)$ if some geodesic ray $[o, \eta)$ intersects a ball of bounded radius around ξ_t (see Lemma 2.8).

Given a group Π of isometries of a metric space (X, d), we define its *critical exponent* as

$$\delta_\Pi := \limsup_{t\to\infty} \frac{1}{t} \log \#\{g \in \Pi \;:\; d(o, go) \leq t\}$$

and its *critical power* as

$$a_\Pi := \limsup_{t\to\infty} \frac{\log \#\{g \in \Pi \;:\; d(o, go) \leq t\} - \delta_\Pi t}{\log t}.$$

Our main result is the following.

Theorem 1.2. *Let Ω be a strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary and $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$ a discrete geometrically finite group which preserves Ω. Assume the convex hull of the limit set C_Γ is hyperbolic with respect to the Hilbert metric, let $o \in C_\Gamma$ be a basepoint, and let μ_o be the Patterson–Sullivan measure. Then there exists a constant C for which the following holds: for any $\xi \in \Lambda_\Gamma$, we have*

$$C^{-1} d(\xi_t, \Gamma o)^{a_\Pi} e^{-\delta_\Gamma t + (2\delta_\Pi - \delta_\Gamma) d(\xi_t, \Gamma o)} \leq \mu_o(V(o, \xi, t)) \leq C d(\xi_t, \Gamma o)^{a_\Pi} e^{-\delta_\Gamma t + (2\delta_\Pi - \delta_\Gamma) d(\xi_t, \Gamma o)}$$

for any $t > 0$, where $\Pi = \{\mathrm{id}\}$ if ξ_t lies in the non-cuspidal part, and otherwise is equal to the stabilizer of the boundary point of the horoball containing ξ_t.

In fact, Theorem 1.2 holds in greater generality, for any *conformal density* of dimension δ, as long as δ is larger than the critical exponent of any parabolic subgroup. We will see a more general result in Theorem 1.4. As in the classical case, Theorem 1.2 implies:

Corollary 1.3. *The Patterson–Sullivan measure μ_o is doubling; that is, there exists $C > 0$ such that for any $\xi \in \Lambda_\Gamma$ and any $r > 0$ we have*

$$\mu_o(D(\xi, 2r)) \leq C \mu_o(D(\xi, r)),$$

where $D(\xi, r)$ denotes the ball of center ξ and radius r for the Gromov metric on the limit set Λ_Γ.

The Gromov metric is defined in Equation (2.2); see also Lemma 2.13 for a comparison between balls in the Gromov metric and shadows of horoballs.

1.1. **Shadow lemma for hyperbolic metric spaces.** In fact, Theorem 1.2 is the consequence of a more general theorem, that we prove for a large class of (Gromov) hyperbolic metric spaces.

Let (X,d) be a Gromov hyperbolic metric space, and let ∂X be its Gromov boundary: for background, see Section 2. If Γ is a geometrically finite group of isometries of X, there is a quasi-invariant horoball decomposition (Proposition 3.3), and there are finitely many Γ-orbits of parabolic points in ∂X. We pick for each such orbit a parabolic point p_i, let Π_i be its stabilizer, and define the function $B_i(t) := \#\{g \in \Pi_i \ : \ d(o, go) \leq t\}$ for any $t \geq 0$ and for a fixed basepoint $o \in X$. Moreover, we define the function $b : X \to \mathbb{R}$ as follows: for $x \in X$, let $b(x) := B_i(2d(x, \Gamma o))$ if x lies in a horoball whose boundary point belongs to Γp_i, and $b(x) := 1$ if x lies in the non-cuspidal part, i.e. it does not belong to any horoball.

The main result in full generality, that we will prove as Theorem 5.1, is:

Theorem 1.4. *Let (X,d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Let μ be a quasi-conformal density of dimension δ on Λ_Γ with no atoms, and assume that Γ has δ-tempered parabolic subgroups. Then there exists a constant C such that for all $\xi \in \Lambda_\Gamma$ and all $t \geq 0$, we have*

$$C^{-1} b(\xi_t) e^{-\delta(t + d(\xi_t, \Gamma o))} \leq \mu(V(o, \xi, t)) \leq C b(\xi_t) e^{-\delta(t + d(\xi_t, \Gamma o))}.$$

For the definition of δ-tempered parabolic subgroups, see Section 3.2. The statement directly generalizes the main theorem of [SV95] for hyperbolic manifolds, and of [Sch04] for Riemannian manifolds with non-constant negative curvature.

1.2. **Dirichlet Theorem.** To state the new result, let $\mathcal{P}$ be the set of parabolic points, and recall that a *horoball* of center $p \in \partial X$ and of radius r is defined as $H_p(r) := \{x \in X : \beta_p(x, o) \leq \log r\}$ where $\beta_p(\cdot, \cdot)$ is the Busemann function at p (see Section 2.3). Given a quasi-invariant horoball decomposition, for each parabolic point p there is a unique horoball H_p centered at p, and we denote the radius of H_p by r_p. Finally, let $\mathcal{H}_p(s)$ be the shadow of the horoball centered at p with radius s, and $\mathcal{P}_s := \{p \in \mathcal{P} \mid r_p \geq s\}$. The following will be proven as Theorem 6.1.

Theorem 1.5 (Dirichlet-type Theorem)**.** *Let (X,d) be a hyperbolic metric space and Γ a geometrically finite group of isometries of X. Then there exist constants $c_1 > 0, c_2 \geq 1$ such that for all $s \leq c_1$, the union*

$$\bigcup_{p \in \mathcal{P}_s} \mathcal{H}_p(c_1 \sqrt{s r_p})$$

covers the limit set Λ_Γ, and there exists $0 < c_3 \leq 1$ such that the shadows $\{\mathcal{H}_p(c_3 \sqrt{s r_p})\}_{p \in \mathcal{P}_s}$ are pairwise disjoint.

We can see this is a Dirichlet-type theorem by considering the classical case of $\mathrm{SL}(2, \mathbb{Z})$ acting on the hyperbolic plane $\mathbb{H}^2$, where the horoballs in the standard horoball packing are centered at rational points $\frac{p}{q}$ with radii $\frac{1}{q^2}$.

1.3. **Applications.** As an application of the shadow lemma (Theorem 1.2) and the Dirichlet theorem (Theorem 1.5), we prove a horoball counting theorem (Proposition 6.4), and a Khinchin-type theorem (Theorem 6.8), culminating in a version of Sullivan's logarithm law for geodesics in the setting of Hilbert geometries:

Theorem 1.6 (Logarithm Law)**.** *Let Ω be a strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary and $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$ a geometrically finite group which preserves Ω. Assume the convex hull C_Γ of* *the limit set is hyperbolic with respect to the Hilbert metric. Let $o \in \Omega$ and let μ_o be the associated Patterson–Sullivan measure. Then for μ_o-almost every ξ in the limit set Λ_Γ, the following holds:*

$$\limsup_{t \to +\infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \frac{1}{2(\delta_\Gamma - \delta_{\max})} \tag{1.1}$$

where $\delta_{\max}$ is the maximal growth rate of any parabolic subgroup.

Intuitively, the logarithm law shows that a generic geodesic makes larger and larger excursions into the cusp of the quotient manifold as time goes by; however, note also that the lim inf in Equation (1.1) is almost surely zero, as almost every geodesic is recurrent to the non-cuspidal part.

In fact, we only use that the space C_Γ is a hyperbolic metric space and the measure satisfies a shadow lemma. More precisely, in Theorem 6.10 we will prove a general logarithm law that applies to any hyperbolic metric space (including e.g. Riemannian manifolds with pinched negative curvature), under the assumption that parabolic subgroups satisfy the *δ-tempered* and *mixed exponential growth* conditions from Definitions 3.4 and 3.6. A logarithm law for Riemannian manifolds of variable negative curvature appears in [HP04] assuming that parabolic subgroups have pure exponential growth. Our Theorem 6.10 generalizes their result to mixed exponential growth.

In a different vein, we also obtain as a consequence of the shadow lemma:

Corollary 1.7 (Singularity with harmonic measure)**.** *Let Ω be a strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary and $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$ a geometrically finite group which preserves Ω. Assume the convex hull of the limit set C_Γ is hyperbolic with respect to the Hilbert metric. Let μ be a measure on Γ with finite superexponential moment, and let ν be the hitting measure of the random walk driven by μ. If Γ contains at least one parabolic element, then ν is singular with respect to the Patterson–Sullivan measure.*

1.4. **Historical remarks.** The global shadow lemma and logarithm law are originally due to Sullivan in the constant negative curvature, finite volume setting [Sul79, Sul84, Sul82]. The argument was generalized and expanded upon to the geometrically finite setting by Stratmann-Velani [SV95], and to the Riemannian setting of variable negative curvature by Hersonsky-Paulin [HP02, HP04, HP07, HP10] and Paulin-Pollicott [PP16]. Schapira earlier proved the global shadow lemma in the Riemannian setting under certain growth conditions on the parabolic subgroups [Sch04]. In a different direction, influential work of Kleinbock-Margulis extends Sullivan's logarithm law to non-compact Riemannian symmetric spaces [KM98, KM99]. Fishman–Simmons–Urbański prove a different version of a Dirichlet theorem and a Khinchin-type theorem in the setting of hyperbolic metric spaces [FSU18]. We point the interested reader to a survey of Athreya for more historical context [Ath09].

The dynamics of the Hilbert geodesic flow was first studied by Benoist in the cocompact setting. For the cocompact case, Benoist proved that the Anosov property of the Hilbert geodesic flow, strict convexity of Ω, C^1-regularity of the boundary, and hyperbolicity of the Hilbert metric are all equivalent [Ben04, Théorème 1.1]. More recently, [CLT15, Theorem 0.15, Theorem 11.6] generalized this result to the non-compact, finite volume case (without the Anosov property, which does not apply to a non-compact phase space). Finally, [CM14a] introduced and studied two definitions of geometrically finite action in Hilbert geometry, which were then studied also by Blayac and Zhu [Bla21, Zhu20, BZ23]. Note that the paper [CM14a] contains mistakes, which however do not affect the results of this paper. We discuss the connections between our work and that of Crampon–Marquis and Blayac–Zhu in Section 7.1. Crampon, Marquis, Blayac, and Zhu study Patterson–Sullivan measures in the geometrically finite setting [Cra11, CM14b, Bla21, Zhu20, BZ23]. We discuss their work in Section 7.3.

For hyperbolic groups, a version of the shadow lemma for the Patterson–Sullivan measure associated to the word metric is proven by Coornaert [Coo93]. This has been more recently generalized by Yang for relatively hyperbolic groups [Yan21].

1.5. **Structure of the paper.** In Section 2, we recall some background material on hyperbolic metric spaces and establish some properties of horoballs and projections that we will need later. In Section 3, we define the notions of geometrical finiteness and δ-tempered parabolic subgroups that we use. In Sections 4 and 5 we prove the main result, Theorem 1.4, for general hyperbolic metric spaces. The Dirichlet Theorem (Theorem 1.5) and the applications in hyperbolic metric spaces are addressed in Section 6, including the logarithm law (Theorem 1.6). Finally, in Section 7 we discuss the applications to Hilbert geometry, completing the proof of Theorem 1.2.

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2. Hyperbolic metric spaces

In this section, we discuss properties of a general hyperbolic metric space (X, d), which we will apply to the Hilbert metric in later sections. Most results should be well-known to experts, but we report them here in the precise form we need them.

2.1. **Gromov product and inner triangle.** Let (X, d) be a geodesic metric space. Given $x, y \in X$, we denote as $[x, y]$ a choice of geodesic segment with endpoints x and y. Note that X needs not be uniquely geodesic, so there may be more than one choice, but for all our statements it will not matter.

Now, consider a geodesic triangle with vertices $x, y, z \in X$ and sides $[x, y]$, $[y, z]$ and $[x, z]$. Then there exist three points $a \in [y, z]$, $b \in [x, z]$, $c \in [x, y]$ such that $d(x, b) = d(x, c)$, $d(y, a) = d(y, c)$, $d(z, a) = d(z, b)$.

We define the *Gromov product* of y, z centered at x as

$$\langle y, z\rangle_x := \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

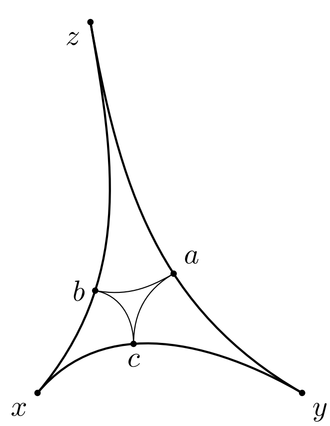

FIGURE 1. Inner triangles in Gromov hyperbolic metric spaces. The point b is such that $\langle y, z\rangle_x = d(x, b) = d(x, c)$.

In the above notation, $\langle y, z\rangle_x = d(x,b) = d(x,c)$. We call the triangle with vertices $\{a,b,c\}$ the *inner triangle* associated to the points x,y,z, and denote it as $\Delta(x,y,z)$. A geodesic metric space is *Gromov hyperbolic* (from now on, simply *hyperbolic*) if there exists a constant α such that for any $x,y,z \in X$, the inner triangle $\Delta(x,y,z)$ has diameter at most α. We denote as $O(\alpha)$ a quantity which depends only on the hyperbolicity constant α.

2.2. **Busemann functions.** Given $z \in X$, we define the *Busemann function* $\beta_z : X \times X \to \mathbb{R}$ as

$$\beta_z(x,y) := d(x,z) - d(y,z).$$

Note that level sets of the Busemann functions are metric spheres centered at z. For each z, the Busemann function $\beta_z(\cdot,\cdot)$ is anti-symmetric, 1-Lipschitz with respect to the L^1 metric on $X \times X$, and equivariant for any group of isometries of X. Moreover, the Busemann function is a cocycle, meaning for $x,y,z,w \in X$,

$$\beta_z(x,y) = \beta_z(x,w) + \beta_z(w,y).$$

Moreover, it satisfies

$$2\langle y,z\rangle_x = \beta_y(x,p) + \beta_z(x,p)$$

for any $p \in [y,z]$.

2.3. **Extension to the boundary and horoballs.** We denote as ∂X the *Gromov boundary* or *hyperbolic boundary* of X, that is (if X is proper) the set of geodesic rays from a given basepoint o, where we identify rays which lie within bounded distance of each other. If X is not proper, the definition of ∂X is a bit more involved (see e.g. [BH99, Def. III.H.3.12]), but in our applications we will focus only on the proper case.

In a hyperbolic space, Gromov products extend coarsely to the hyperbolic boundary, by setting for any $o \in X$, $\xi, \eta \in \partial X$

$$\langle \xi, \eta\rangle_o := \liminf_{\substack{y\to\xi \\ z\to\eta}} \langle y,z\rangle_o.$$

Similarly, for $\xi \in \partial X$, $x, y \in X$, one defines the *Busemann function* as

$$\beta_\xi(x,y) := \liminf_{z\to\xi} \beta_z(x,y).$$

These extensions are coarsely well-defined, meaning that

$$|\liminf_{\substack{y\to\xi \\ z\to\eta}} \langle y,z\rangle_x - \limsup_{\substack{y\to\xi \\ z\to\eta}} \langle y,z\rangle_x| \le O(\alpha), \qquad |\liminf_{z\to\xi} \beta_z(x,y) - \limsup_{z\to\xi} \beta_z(x,y)| \le O(\alpha). \tag{2.1}$$

It follows from Equation 2.1 that the chosen definition of Busemann function is a *quasi-cocycle*, meaning for $\xi \in \partial X$, $x,y,z \in X$

$$\beta_\xi(x,y) = \beta_\xi(x,z) + \beta_\xi(z,y) + O(\alpha).$$

The Busemann functions are also anti-symmetric, meaning $\beta_\xi(x,y) = -\beta_\xi(y,x)$. Lastly, note that taking the liminf allows us to conclude that Busemann functions are isometry-invariant, meaning: for any isometry g of (X,d) and $\xi \in \partial X$, $x, y \in X$,

$$\beta_{g\xi}(gx, gy) = \beta_\xi(x,y).$$

Also, as usual, these Busemann functions are 1-Lipschitz by the triangle inequality.

The notion of Busemann function allows us to extend the definition of the inner triangle to the boundary. Namely, for $x,y,z \in X \cup \partial X$, there exist three points $a \in [y,z]$, $b \in [x,z]$, $c \in [x,y]$ such that $\beta_x(b,c) = \beta_y(a,c) = \beta_z(a,b) = 0$. We say a,b,c are the vertices of the inner triangle $\Delta(x,y,z)$, and note that they are again within distance $O(\alpha)$. Note that this definition agrees with the definition of inner triangle when $x,y,z \in X$.

A horoball is a sublevel set of the Busemann function. More precisely, given $\xi \in \partial X$ and $r > 0$, the *horoball centered at* ξ *of radius* r is

$$H_\xi(r) := \{x \in X : \beta_\xi(x, o) \leq \log r\}.$$

The *horosphere centered at* ξ *of radius* r is the set where equality holds.

2.4. **Projections.** The notion of closest point projection will be fundamental in our paper.

Given a point $x \in X$ and a geodesic $[y, z]$, a point $p \in [y, z]$ is a *closest point projection* of x onto $[y, z]$ if it minimizes its distance to x: that is, $d(x, p) \leq d(x, q)$ for any $q \in [y, z]$, or equivalently, $\beta_x(p, q) \leq 0$ for any $q \in [y, z]$. Similarly, for $x, y, z \in X \cup \partial X$, closest point projection of x onto $[y, z]$ is any point p such that $\beta_x(p, q) \leq 0$ for all $q \in [y, z]$.

Closest point projection is not unique, but, in hyperbolic metric spaces, it is well-defined up to bounded distance: in fact, any closest point projection of x onto $[y, z]$ lies within distance $O(\alpha)$ of any point of the inner triangle $\Delta(x, y, z)$. Hence, any two closest point projections lie within distance $O(\alpha)$ of each other.

To see this, first recall that hyperbolic metric spaces satisfy the reverse triangle inequality:

Proposition 2.1 (see e.g. [MT18], Proposition 2.2)**.** *Let* (X, d) *be a hyperbolic metric space, let* γ *be a geodesic in* X*,* $y \in X$ *a point, and* q *a closest point projection of* y *to* γ*. Then for any* $z \in \gamma$*,*

$$d(y, z) = d(y, q) + d(z, q) + O(\alpha).$$

Then the following lemma implies, for instance, that closest point projection is coarsely well-defined:

Lemma 2.2. *Let* (X, d) *be a hyperbolic metric space, let* $o \in X$*,* $\eta, \xi \in X \cup \partial X$*, and let* p *be a closest point projection of* η *onto* $[o, \xi)$*. Then*

$$\langle \eta, \xi \rangle_o = d(o, p) + O(\alpha).$$

Consequently, p *is within* $O(\alpha)$ *of the inner triangle* $\Delta(\eta, o, \xi)$*.*

Proof. Let us first suppose that $\xi, \eta \in X$. Then by the reverse triangle inequality in Proposition 2.1,

$$\begin{aligned} d(o, \eta) &= d(o, p) + d(p, \eta) + O(\alpha) \\ d(\xi, \eta) &= d(p, \xi) + d(p, \eta) + O(\alpha) \end{aligned}$$

hence

$$\begin{aligned} 2\langle \eta, \xi \rangle_o &= d(o, \eta) + d(o, \xi) - d(\eta, \xi) \\ &= d(o, p) + d(p, \eta) + d(o, p) + d(p, \xi) - d(p, \xi) + d(p, \eta) + O(\alpha) \\ &= 2d(o, p) + O(\alpha). \end{aligned}$$

The claim then follows letting ξ, η go to the boundary. □

We now look at closest point projection onto horoballs.

Lemma 2.3 (Horoball projection)**.** *Let* (X, d) *be a hyperbolic metric space and fix* $o \in X$*. Then for all horoballs* H *centered at* $\xi \in \partial X$ *and not containing* o*, geodesic rays* $[o, \xi)$*,* $p \in [o, \xi) \cap \partial H$*, and* $x \in [o, \xi)$*,*

$$d(x, \partial H) \leq d(x, p) \leq d(x, \partial H) + O(\alpha).$$

Proof. Let $q \in \partial H$. First, consider $x \notin H$. Then by definition, $\beta_\xi(q, o) = \beta_\xi(p, o)$, hence by the quasi-cocycle property, $\beta_\xi(p, q) = O(\alpha)$. Let $z_n \in [o, \xi)$ be a sequence converging to ξ. Then for each n sufficiently large,

$$d(x, p) + d(p, z_n) = d(x, z_n) \leq d(x, q) + d(q, z_n)$$

hence by definition of β_{z_n} and choice of z_n,

$$O(\alpha) = \beta_{z_n}(p,q) \leq d(x,q) - d(x,p).$$

Hence, $d(x,p) \leq d(x,q) + O(\alpha)$.

Now, assume $x \in H$. Then $\beta_\xi(q,x) = \beta_\xi(p,x) + O(\alpha)$ by the quasi-cocycle property, and similarly

$$d(x,q) \geq |\beta_\xi(q,x)| = |\beta_\xi(p,x)| + O(\alpha) = d(x,p) + O(\alpha)$$

which concludes the proof. □

Using the notation in Lemma 2.3, $\beta_\xi(p,o) = \log r$, hence:

Corollary 2.4. *For H a horoball of radius r, we have*

$$\log r = -d(o,H) + O(\alpha).$$

2.5. **Shadows.** We can now introduce the definition of shadow.

Definition 2.5. Let (X,d) be a hyperbolic metric space, $o \in X$ and $\xi \in X \cup \partial X$. The *shadow* from o to ξ of depth $t \geq 0$ is the set

$$V(o,\xi,t) := \{\eta \in \partial X \ : \ \langle \eta, \xi \rangle_o \geq t\}.$$

Note that, for any isometry g of X,

$$gV(o,\xi,t) = V(go,g\xi,t).$$

In a hyperbolic metric space X, shadows of varying depth generate the topology on ∂X.

2.6. **Fellow traveling.** In a hyperbolic metric space, geodesic rays converging to the same boundary point satisfy strong fellow traveling properties.

Lemma 2.6 (Asymptotic geodesics in a hyperbolic metric space)**.** *Let (X,d) be a hyperbolic metric space. Fix $\xi \in X \cup \partial X$ and $x,y \in X$ and denote by x_t, y_t the points on geodesic rays $[x,\xi), [y,\xi)$ which are distance t from x and y, respectively. Then for all $0 < t \leq \min\{d(x,\xi), d(y,\xi)\}$,*

$$d(x_t,y_t) \leq d(x,y) + O(\alpha).$$

Proof. Let p be a closest point projection of ξ onto $[x,y]$, and suppose by symmetry that $d(x,p) \leq d(y,p)$. If $t < d(x,p)$ then $d(x_t,y_t) = d(x,y) - 2t + O(\alpha)$.

If $d(x,p) \leq t < d(y,p)$, then $d(x_t,p) = t - d(x,p) + O(\alpha)$ and $d(y_t,p) = d(p,y) - t + O(\alpha)$ so $d(x_t,y_t) \leq d(x_t,p) + d(p,y_t) = d(y,p) - d(x,p) + O(\alpha) \leq d(x,y) + O(\alpha)$.

If $t \geq d(y,p)$ then $d(x_t,p) = t - d(x,p) + O(\alpha)$ and $d(y_t,p) = t - d(y,p) + O(\alpha)$, so $d(x_t,y_t) = |d(x_t,p) - d(y_t,p)| + O(\alpha) = |d(x,p) - d(y,p)| + O(\alpha) \leq d(x,y) + O(\alpha)$. □

Given three points $x,y,z \in X \cup \partial X$, we say that two points $p \in [x,y]$ and $q \in [x,z]$ are *comparable* if $\beta_x(p,q) = 0$ and $\beta_x(a,p) \geq 0$ where $a \in [x,y]$ is a vertex of the inner triangle $\Delta(x,y,z)$. Lemma 2.6 and the definition of inner triangle immediately implies:

Corollary 2.7. *Let (X,d) be a hyperbolic metric space and for any $x,y,z \in X \cup \partial X$, let $p \in [x,y]$, $q \in [x,z]$ be comparable points. Then $d(p,q) \leq O(\alpha)$.*

The next lemma follows the preceding two lemmas:

Lemma 2.8 (Fellow traveling)**.** *Let (X,d) be a hyperbolic metric space, let $x,y \in X$ and ξ,η in ∂X. Denote by ξ_t, η_t geodesic rays from x to ξ and from y to η, respectively, parameterized at unit speed. If $\eta \in V(x,\xi,t)$ then $d(\xi_s,\eta_s) \leq d(x,y) + O(\alpha)$ for all $s \in [0,t]$.*

Proof. Since η is in $V(x, \xi, t)$, by Lemma 2.2 any closest point projection of η onto (x, ξ) is distance greater than $t + O(\alpha)$ from x. Let q be the point on a geodesic ray (x, η) which is distance t from x. Then (up to $O(\alpha)$), q and ξ_t are comparable points on the thin triangle with vertices x, ξ, and η, and thus by Corollary 2.7 their distance is bounded above by $O(\alpha)$. On the other hand, the distance from q to η_t is bounded above by $d(x, y)$ by Lemma 2.6. The conclusion follows from the triangle inequality. ☐

2.7. **Projections and Busemann functions.** An immediate corollary of Proposition 2.1 is:

Corollary 2.9. *Let (X, d) be a hyperbolic metric space, γ a geodesic in X, $y \in X \cup \partial X$ a point, and q a closest point projection of y to γ. Then for any $z \in \gamma$,*

$$\beta_y(z, q) = d(z, q) + O(\alpha).$$

The next lemma readily follows from Corollary 2.9.

Lemma 2.10. *Let (X, d) be a hyperbolic metric space, let γ be a (finite or infinite) geodesic, let $\eta \in X \cup (\partial X \setminus \overline{\gamma})$ and let p be a closest point projection of η to γ. Then for any $x, y \in \gamma$ we have*

$$\beta_\eta(x, y) = \beta_p(x, y) + O(\alpha).$$

Proof. By the cocycle property

$$\beta_\eta(x, y) = \beta_\eta(x, p) - \beta_\eta(y, p) + O(\alpha)$$

and using Corollary 2.9

$$\begin{aligned} &= d(x, p) - d(y, p) + O(\alpha) \\ &= \beta_p(x, y) + O(\alpha). \end{aligned}$$

The equality then also holds for $\eta \in \partial X \setminus \overline{\gamma}$ as the closest point projection extends coarsely continuously. ☐

Lemma 2.11. *Let (X, d) be a hyperbolic metric space, $o \in X$ a basepoint, $\xi \in \partial X$ and ξ_t the point on a geodesic ray $[o, \xi)$ at distance t from o. If $\eta \in V(o, \xi, t)$, then*

$$\beta_\eta(o, \xi_t) = t + O(\alpha).$$

On the other hand, if $\eta \notin V(o, \xi, D)$, then

$$-t \leq \beta_\eta(o, \xi_t) \leq -t + 2D + O(\alpha).$$

Proof. Let p be a closest point projection of η onto $[o, \xi)$. Since $\eta \in V(o, \xi, t)$ and by Lemma 2.2, p lies between $\xi_{t-O(\alpha)}$ and ξ. Then by Lemma 2.10

$$\beta_\eta(o, \xi_t) = \beta_p(o, \xi_t) + O(\alpha) = t + O(\alpha).$$

To prove the second part, if $\eta \notin V(o, \xi, D)$, then $d(o, p) \leq D$, so

$$\beta_p(o, \xi_t) = -t + 2d(o, p) \leq -t + 2D$$

so the upper bound follows from Lemma 2.10. The lower bound follows from the triangle inequality. ☐

2.8. **Shadows in hyperbolic spaces.** We will now state two lemmas on hyperbolic metric spaces and shadows that we will need later.

Lemma 2.12. *Let (X, d) be a hyperbolic metric space, $o \in X$ a basepoint, $x, y \in X$, and $\xi \in X \cup \partial X$.*

(1) If $\eta \in V(o, \xi, t)$, then

$$V(o, \eta, t + O(\alpha)) \subseteq V(o, \xi, t) \subseteq V(o, \eta, t - O(\alpha)).$$

(2) For all $M > 0$, there is a constant $A > 0$ such that if $d(x, y) \leq M$, then for all $\xi \in \partial X$ and all $t > 0$,

$$V(x, \xi, t + A) \subset V(y, \xi, t) \subset V(x, \xi, t - A).$$

In a hyperbolic metric space (X, d), there is a metric $d_{\partial X}$ on ∂X called the *Gromov metric* with the property that

$$c^{-1} e^{-\epsilon\langle \xi, \eta \rangle_o} \leq d_{\partial X}(\xi, \eta) \leq c e^{-\epsilon\langle \xi, \eta \rangle_o} \tag{2.2}$$

for some uniform constant c and $\epsilon > 0$, and any $\eta, \xi \in \partial X$. We refer the reader to [BH99, Prop. III.H.3.21] for this result and additional background.

Given a basepoint $o \in X$, we now define the *shadow of a set* to be the set of all endpoints $\xi \in \partial X$ of geodesic rays starting from o which intersect the set. The *shadow of a horoball* centered at $\xi \in \partial X$ of radius r is denoted $\mathcal{H}_\xi(r)$.

Lemma 2.13. *Let (X, d) be a hyperbolic metric space. Then there exists a constant C such that for all $\xi \in \partial X$ and $r > 0$, the shadow of a horoball $\mathcal{H}_\xi(r)$ has diameter s in the Gromov metric, where $C^{-1} r^\epsilon \leq s \leq C r^\epsilon$.*

Proof. Let ξ be the boundary point of the horoball $H = H_\xi(r)$. Let p be a closest point projection of ξ onto $[o, \eta)$. By Corollary 2.9 and Lemma 2.2,

$$\beta_\xi(o, p) = d(o, p) + O(\alpha) = \langle \xi, \eta \rangle_o + O(\alpha).$$

Let $q \in [o, \xi) \cap \partial H$. Then by Lemma 2.3 and Corollary 2.4,

$$\beta_\xi(o, q) = d(o, H) + O(\alpha) = -\log r + O(\alpha).$$

Since p minimizes $\beta_\xi(x, o)$ for all $x \in [o, \eta)$ by definition, if $\eta \in \mathcal{H}_\xi(r)$, then $p \in H$ hence $[o, \eta) \cap H \neq \varnothing$ and $\beta_\xi(o, q) \geq \beta_\xi(o, p)$. Thus,

$$d_{\partial X}(\eta, \xi) \leq c e^{-\epsilon\langle \xi, \eta \rangle_o} \leq c e^{-\epsilon \beta_\xi(o,p) + O(\alpha)} \leq c e^{-\epsilon \beta_\xi(o,q) + O(\alpha)} \leq c e^{O(\alpha)} r^\epsilon.$$

Analogously, if $\eta \notin \mathcal{H}_\xi(r)$, then $\beta_\xi(o, q) < \beta_\xi(o, p)$, and the lower bound follows. □

Lemma 2.14. *Let (X, d) be a hyperbolic metric space. Then for all $\xi \in \partial X$,*

$$V(o, \xi, -\log r + O(\alpha)) \subset \mathcal{H}_\xi(r) \subset V(o, \xi, -\log r - O(\alpha)).$$

Proof. The proof follows from Lemma 2.13 and Equation 2.2. □

2.9. **Disjointness.** The following lemmas will be used in the proof of Theorem 6.1.

Lemma 2.15. *Let (X, d) be a hyperbolic metric space, with $o \in X$ and $\xi_1, \xi_2 \in \partial X$ with $\xi_1 \neq \xi_2$. Let $q_1 \in [o, \xi_1)$ and $q_2 \in [o, \xi_2)$ with $\beta_{\xi_1}(o, q_2) \geq \beta_{\xi_1}(o, q_1)$. Then there exists $z \in [o, \xi_2)$ such that*

$$\beta_{\xi_1}(o, z) \geq \frac{d(o, q_1) + d(o, q_2)}{2} - O(\alpha).$$

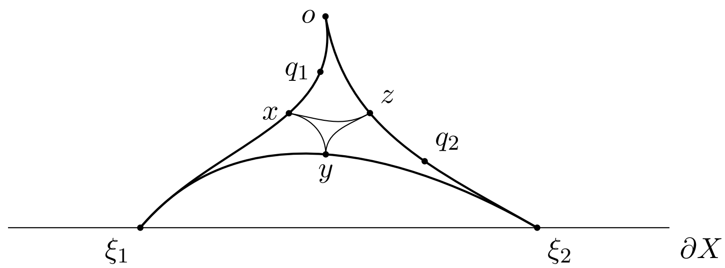

FIGURE 2. An approximate tree for the proof of Lemma 2.15.

Proof. By hyperbolicity, the triangle $[o, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, o]$ is thin. Let x, y, z be the vertices of its inner triangle, with $x \in [o, \xi_1)$, $y \in (\xi_1, \xi_2)$, $z \in [o, \xi_2)$. By Lemma 2.2, z is within $O(\alpha)$ of any closest point projection of ξ_1 onto $[o, \xi_2)$, so by Corollary 2.9,

$$\begin{aligned}\beta_{\xi_1}(o, q_2) &= \beta_{\xi_1}(o, z) - \beta_{\xi_1}(q_2, z) + O(\alpha)\\ &= d(o, z) - d(q_2, z) + O(\alpha).\end{aligned}$$

Moreover, since q_1 lies on $[o, \xi_1)$,

$$\beta_{\xi_1}(o, q_1) = d(o, q_1).$$

Hence, from $\beta_{\xi_1}(o, q_2) \geq \beta_{\xi_1}(o, q_1)$ we obtain

$$0 \leq d(z, q_2) \leq d(o, z) - d(o, q_1) + O(\alpha) = d(o, x) - d(o, q_1) + O(\alpha)$$

which implies either $q_1 \in [o, x]$ or is distance $O(\alpha)$ from x and hence from z. In either case,

$$d(o, z) = d(o, q_1) + d(q_1, z) + O(\alpha).$$

It follows that

$$d(z, q_2) \leq d(z, q_1) + O(\alpha).$$

Moreover,

$$\begin{aligned}d(o, q_2) &\leq d(o, q_1) + d(q_1, z) + d(z, q_2)\\ &\leq d(o, q_1) + 2d(q_1, z) + O(\alpha)\end{aligned}$$

hence

$$\begin{aligned}\frac{d(o, q_2) + d(o, q_1)}{2} &\leq d(o, q_1) + d(q_1, z) + O(\alpha)\\ &= \beta_{\xi_1}(o, z) + O(\alpha)\end{aligned}$$

which proves the claim. □

We will see later that a key property of our set-up, as in Sullivan's original one [Sul82], is that horoballs are disjoint. This has the following consequences.

Lemma 2.16. *Let (X, d) be a hyperbolic metric space. Let $\xi_1, \xi_2 \in \partial X$ and let H_1, H_2 be horoballs based at ξ_1, ξ_2. Define as q_i an intersection point of ∂H_i and $[o, \xi_i)$ for $i = 1, 2$. If $H_1 \cap H_2 = \emptyset$, then*

$$\langle \xi_1, \xi_2 \rangle_o \leq \frac{d(o, q_1) + d(o, q_2)}{2} + O(\alpha).$$

Proof. By symmetry, let us assume that $d(o, q_1) \leq d(o, q_2)$. Let x be a closest point projection of ξ_2 onto $[o, \xi_1)$ and z a closest point projection of ξ_1 onto $[o, \xi_2)$. Then x and z are within distance $O(\alpha)$ by Lemma 2.2.

If $q_1 \in [x, \xi_1)$, then $d(o, q_2) \geq d(o, q_1) \geq d(o, x) = \langle \xi_1, \xi_2 \rangle_o$, hence the claim is trivially true.

Suppose $q_1 \in [o, x]$. See Figure 3. Since H_1 and H_2 are disjoint, then q_2 does not belong to H_1, hence $\beta_{\xi_1}(z, q_2) < \beta_{\xi_1}(z, q_1) + O(\alpha)$. By Corollary 2.9, $\beta_{\xi_1}(q_2, z) = d(q_2, z) + O(\alpha)$. Noting that q_1 is within distance $O(\alpha)$ of $[o, z]$ by Corollary 2.7 gives similarly that $\beta_{\xi_1}(q_1, z) = d(q_1, z) + O(\alpha)$, hence $d(x, q_2) \geq d(x, q_1) + O(\alpha)$. Then from

$$d(q_1, x) = d(o, x) - d(o, q_1)$$
$$d(q_2, x) = d(o, q_2) - d(o, x) + O(\alpha)$$

we obtain

$$\langle \xi_1, \xi_2 \rangle_o = d(o, x) \leq \frac{d(o, q_1) + d(o, q_2)}{2} + O(\alpha).$$

□

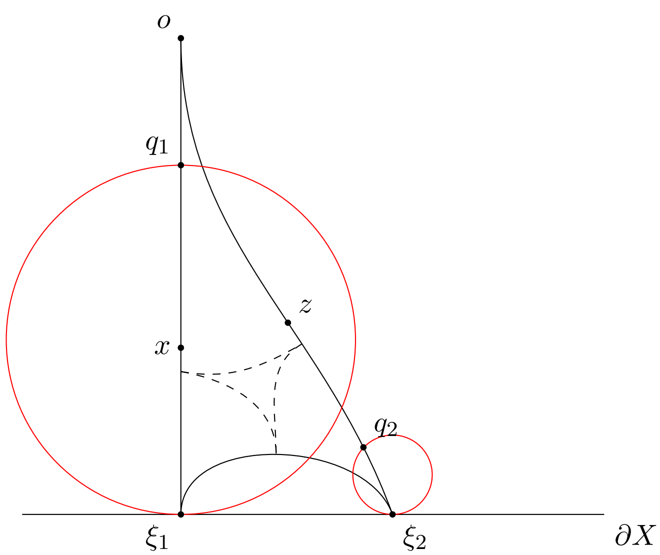

FIGURE 3. For the proof of Lemma 2.16, in the case that $q_1 \in [o, x]$. Note that x and z are within $O(\alpha)$ of the inner triangle $\Delta(o, \xi_1, \xi_2)$.

The next corollary now follows from Equation (2.2) and Lemma 2.16.

Corollary 2.17. *Let (X, d) be a hyperbolic metric space, $\xi_1, \xi_2 \in \partial X$, and $r_1, r_2 > 0$. Then there exists a constant $C > 0$ such that, if the horoballs $H_{\xi_1}(r_1)$ and $H_{\xi_2}(r_2)$ are disjoint, then*

$$d_{\partial X}(\xi_1, \xi_2) \geq C(r_1 r_2)^{\frac{\epsilon}{2}}.$$

3. Geometrical finiteness

Let (X, d) be a proper, geodesic metric space, and Γ a countable group of isometries of X acting properly discontinuously on X. Assume that X has a compactification $\overline{X}$, namely X embeds as an open, dense, subset of a compact metrizable space $\overline{X}$, and the action of Γ extends to an action on $\overline{X}$ by homeomorphisms. The set $\partial_{top} X := \overline{X} \setminus X$ is the *topological boundary* of X. Given a basepoint $o \in X$, define the *limit set* of Γ as

$$\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o.$$

We say that the action of Γ on X is *non-elementary* if $|\Lambda_\Gamma| \geq 3$, and we denote by C_Γ the *convex hull* of Λ_Γ in X. More specifically, C_Γ is the union of all biinfinite geodesics which have both endpoints in Λ_Γ.

Given $\gamma \in \Gamma$, we define its *translation distance* as $\tau(\gamma) := \inf_{x\in X} d(x, \gamma x)$. We define an element γ to be *elliptic* if $\tau(\gamma) = 0$ and the infimum is attained, *parabolic* if $\tau(\gamma) = 0$ and the infimum is not attained, *loxodromic* if $\tau(\gamma) > 0$ and the infimum is attained, and *quasi-loxodromic* if $\tau(\gamma) > 0$ and the infimum is not attained.

A subgroup $\Pi < \Gamma$ is a *parabolic group* if Π has infinite order, fixes a single point of $\partial_{top}X$, and no element of Π is loxodromic. We call $\xi \in \Lambda_\Gamma$ a *parabolic point* if its stabilizer $\mathrm{stab}_\Gamma(\xi)$ is a parabolic subgroup. We say a parabolic point ξ is *bounded parabolic* if the quotient $(\Lambda_\Gamma \setminus \{\xi\})/\mathrm{stab}_\Gamma(\xi)$ is compact. A point $\xi \in \Lambda_\Gamma$ is a *conical limit point* if there exist a sequence $(\gamma_n) \subseteq \Gamma$ and distinct points $a, b \in \Lambda_\Gamma$ such that $\gamma_n\xi \to a$ and $\gamma_n\eta \to b$ for all $\eta \in \Lambda_\Gamma \setminus \{\xi\}$.

Let us from now on assume that the space (X, d) is Gromov hyperbolic; then we can take as $\overline{X}$ its Gromov compactification, and we denote as $\partial X = \partial_{top}X$ its Gromov boundary as in Subsection 2.3. Note that the subset (C_Γ, d) is again a proper geodesic hyperbolic metric space on which Γ acts properly discontinuously.

Let us now assume that Γ is non-elementary; then Gromov showed that Λ_Γ is the smallest closed Γ-invariant subset of ∂X (see e.g. [Coo93, Théorème 5.1]), hence Λ_Γ is basepoint independent. Moreover, every isometry $\gamma \in \Gamma$ is either elliptic, parabolic, or loxodromic; every parabolic element is infinite order and has exactly one fixed point in ∂X, and any loxodromic element is infinite order and has exactly two fixed points in ∂X [Bow99, Lemma 2.1].

We now define geometrical finiteness as Bowditch does [Bow12, p 38], inspired by the work of Beardon-Maskit [BM74, Theorems 2 and 3] on characterizing existence of finite-sided fundamental domains for Kleinian groups:

Definition 3.1. Let (X, d) be a proper, hyperbolic metric space and Γ a non-elementary group of isometries acting properly discontinuously on X. Then Γ is *geometrically finite* if every point of Λ_Γ is either conical or bounded parabolic.

Remark 3.2. We will at times reference the work of Bowditch [Bow12] for geometrically finite groups Γ acting on a hyperbolic metric space (X, d) such that Γ acts on ∂X minimally. Bowditch notes that this framework is general by simply replacing X with C_Γ in any situation where $\Lambda_\Gamma \neq \partial X$, as $\partial C_\Gamma = \Lambda_\Gamma$ follows from the definition.

Let $\mathcal{P}$ be the collection of parabolic fixed points in ∂X for the action of Γ. By [Bow12, Proposition 6.15], if Γ is a geometrically finite group of isometries of X, then there are finitely many orbits of parabolic points in $\mathcal{P}$, hence we may write

$$\mathcal{P} = \bigsqcup_{i=1}^{a} \mathcal{P}^i$$

where each $\mathcal{P}^i$ is the orbit of a parabolic point.

3.1. **Horoball decomposition.** Let $\mathcal{P}$ be the set of parabolic points in Λ_Γ, which we note is Γ-invariant. We define a *quasi-invariant family of horoballs* to be a collection $\{H_p\}_{p\in\mathcal{P}}$ of mutually disjoint horoballs H_p centered at p for which there exists a constant C such that $d(H_{\gamma p}, \gamma H_p) \leq C$ for all $\gamma \in \Gamma, p \in \mathcal{P}$. If in fact $H_{\gamma p} = \gamma H_p$ then $\{H_p\}_{p\in\mathcal{P}}$ is an *invariant family of horoballs.* Such a family is said to be *r-separated* if $d(H_p, H_q) \geq r$ for all $p \neq q \in \mathcal{P}$. Given a quasi-invariant family of horoballs $\{H_p\}_{p\in\mathcal{P}}$, the corresponding *non-cuspidal part* for the action of Γ on X is the set

$$X_{nc} := C_\Gamma \setminus \bigcup_{p\in\mathcal{P}} H_p,$$

and the *cuspidal part* for the action of Γ on X is

$$X_c := \bigcup_{p\in\mathcal{P}} C_\Gamma \cap H_p.$$

The decomposition $C_\Gamma = X_{nc} \cup X_c$ is called a *horoball decomposition of* X or of C_Γ. At the level of quotient, the *non-cuspidal part* is $M_{nc} := X_{nc}/\Gamma$ and the *cuspidal part* is $M_c := X_c/\Gamma = M \smallsetminus X_{nc}$. Similarly, $M = M_{nc} \cup M_c$ is a *horoball decomposition of* M.

Proposition 3.3. *Let (X,d) be a proper hyperbolic metric space, Γ a group of isometries of (X,d) acting properly discontinuously on X. If Γ is geometrically finite, then $\mathcal{P}/\Gamma$ is finite and there exists an r-separated quasi-invariant family of open horoballs $\{H_p\}_{p\in\mathcal{P}}$ centered at each of the parabolic fixed points such that the non-cuspidal part M_{nc} is compact.*

Proof. Bowditch's [Bow12, Proposition 6.13] states the conclusion, but for some more general notion of horoballs arising as sublevel sets of more general horofunctions [Bow12, p 29]: given $p \in \Lambda_\Gamma$, we say $h_p \colon X \to \mathbb{R}$ is a *horofunction centered at* p if for any $x \in X$ and any $a \in X$ that is within distance $O(\alpha)$ of $[x,p)$, then $h_p(a) = h_p(x) + d(x,a) + O(\alpha)$. We will refer to a sublevel set of a horofunction as a *generalized horoball.* To compare to our definition of Busemann function, let a be the vertex of the inner triangle $\Delta(p,x,o)$ on a geodesic ray $[o,p)$. Bowditch's definition of horofunction implies immediately that

$$\begin{aligned} h_p(x) - h_p(o) &= d(o,a) - d(x,a) + O(\alpha) \\ &= \beta_p(o,a) + \beta_p(a,x) + O(\alpha) = \beta_p(o,x) + O(\alpha). \end{aligned}$$

It follows that

$$h_p(x) = \beta_p(o,x) + h_p(o) + O(\alpha).$$

As a consequence, every generalized horoball is within distance $O(\alpha)$ of a horoball. The conclusion now follows [Bow12, Proposition 6.13] which states that there exists an r-separated invariant family of generalized horoballs such that M_{nc} is compact. □

3.2. **Tempered growth.** We say two positive real valued functions f and h are *asymptotically equivalent*, denoted $f \asymp h$, if there exists a uniform constant $k \geq 1$ for which $k^{-1}h \leq f \leq kh$.

Assume (X,d) is a proper hyperbolic metric space and Γ is a geometrically finite group of isometries of X. Fix a basepoint $o \in X$. Recall the *critical exponent* of Γ is

$$\delta_\Gamma := \limsup_{t\to\infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : d(o,\gamma o) \leq t\}.$$

Equivalently, δ_Γ is the infimum over values of s for which the Poincaré series $P_\Gamma(s) := \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}$ converges. Fix a parabolic group Π. Let us denote as

$$B_\Pi(t) := \#\{g \in \Pi \ : \ d(o,go) \leq t\}.$$

Given $r > 0$, we define the *annular growth function*

$$A_{\Pi,r}(t) := \frac{1}{r} \log \left(\frac{B_\Pi(t+r)}{B_\Pi(t)} \right).$$

We define the *lower* and *upper annular growth rates* of Π as, respectively,

$$\delta_\Pi^- := \lim_{r\to\infty} \inf_{t>0} A_{\Pi,r}(t), \qquad \delta_\Pi^+ := \lim_{r\to\infty} \sup_{t>0} A_{\Pi,r}(t). \tag{3.1}$$

Note that by the definitions, the limit as $r \to \infty$ exists for both quantities, and that $\delta_\Pi^- \leq \delta_\Pi \leq \delta_\Pi^+$.

Definition 3.4. We say that the parabolic subgroup Π has δ*-tempered growth* if

$$0 < \delta_\Pi^- \leq \delta_\Pi^+ < \delta.$$

If $\Pi < \Gamma$ has δ_Γ-tempered growth where δ_Γ is the critical exponent of Γ, then we simply say Π has *tempered growth.* If every maximal parabolic subgroup of Γ has (δ-)tempered growth, then we say Γ has *(δ-)tempered parabolic subgroups.*

Remark 3.5. Note the following:

(1) Since $B_\Pi(t)$ is nondecreasing for any parabolic subgroup Π, for any $r \le s$ we have
$$rA_{\Pi,r}(t) \le sA_{\Pi,s}(t) \qquad \text{for any } t \ge 0;$$

(2) Note that, for any $k \ge 1$,
$$A_{\Pi,kr}(t) = \frac{1}{k}\sum_{i=0}^{k-1} A_{\Pi,r}(t+ri)$$
so
$$\inf_{t>0} A_{\Pi,r}(t) \le \inf_{t>0} A_{\Pi,kr}(t) \le \sup_{t>0} A_{\Pi,kr}(t) \le \sup_{t>0} A_{\Pi,r}(t).$$

(3) For fixed $s \ge 0$ and Π a parabolic group, there exists a constant C such that for all $t \ge 0$
$$B_\Pi(t) \le B_\Pi(t+s) \le CB_\Pi(t).$$
Indeed, if $s > 0$, let k be such that $kr > s$. Then by (1), for each Π we have
$$\frac{B_\Pi(t+s)}{B_\Pi(t)} = e^{sA_{\Pi,s}(t)} \le e^{krA_{\Pi,kr}(t)}$$
and $A_{\Pi,kr}(t)$ is bounded above since $\sup_{t>0} A_{\Pi,kr}(t)$ exists. It is straightforward to verify similarly that for $s < 0$ there exists a constant C such that for all $t \ge s$,
$$C^{-1}B_\Pi(t) \le B_\Pi(t+s) \le B_\Pi(t).$$

Definition 3.6. We say a parabolic group Π has *mixed exponential growth* if there exist $\delta_\Pi > 0, a_\Pi \ge 0$ such that
$$B_\Pi(t) \asymp e^{\delta_\Pi t}(t+1)^{a_\Pi} \qquad \text{for any } t \ge 0.$$

A straightforward calculation shows that in this case, $\delta_\Pi^- = \delta_\Pi^+ = \delta_\Pi > 0$. Thus any parabolic group with mixed exponential growth has δ-tempered growth for any $\delta > \delta_\Pi$.

3.3. **A technical lemma.** We end this section with a technical lemma which will be needed in Section 4 but only requires the tools and definitions from hyperbolic metric spaces. The lemma closely resembles an analogous lemma of Schapira written in French [Sch04, Lemme 2.9].

Lemma 3.7. *Let (X,d) be a hyperbolic metric space, $\xi \in \partial X$, $K \subseteq \partial X \setminus \{\xi\}$ compact, $o \in X$. Then there exists $A > 0$ such that the following holds. For every g a parabolic isometry of X fixing ξ, we have $|\langle \xi, g\eta\rangle_o - d(o,go)/2| \le A$. In particular,*
$$gK \subseteq V(o,\xi,d(o,go)/2 + A) \setminus V(o,\xi,d(o,go)/2 - A).$$
Moreover, for any $t > A$ and any $\eta \in K$ we have
$$|\beta_{g\eta}(\xi_t, g\xi_t)| = \max\{2t - d(o,go), 0\} \pm A.$$

Proof of Lemma 3.7. Following the set-up of Schapira; for $\eta \in K$, let y be the point on (ξ,η) which is on the same horosphere at ξ as o. Then y is bounded distance from o for all $\eta \in K$ by compactness of K; let C be an upper bound on $d(o,y)$.

Consider a geodesic triangle with endpoints ξ, o, and $g\eta \in gK$. This triangle has a unique inner triangle with vertices $a \in [o,\xi), b \in [o,g\eta)$, and $c \in (\xi, g\eta)$ such that $\beta_o(a,b) = \beta_\xi(a,c) = \beta_{g\eta}(b,c) = 0$.

We will first compare $d(o,go)$ with $2d(o,a)$ (notice that a of course depends on g). Then we will estimate $2d(o,a)$ to prove the containments of shadows. Then we will prove the estimate on the Busemann functions.

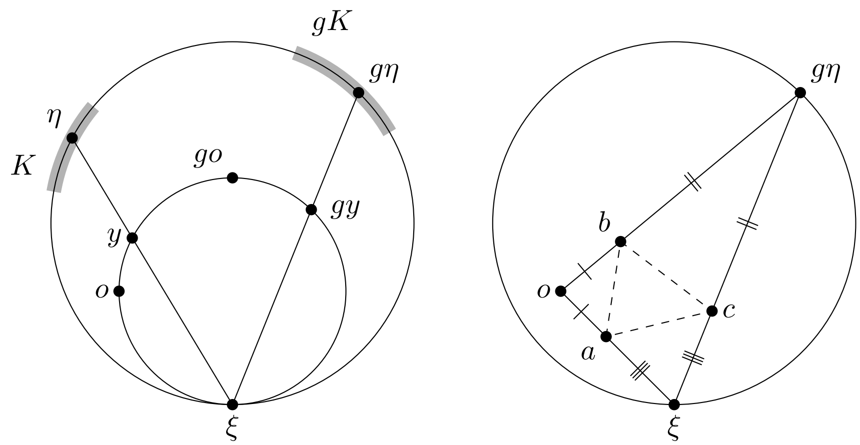

FIGURE 4. The set-up of Lemma 3.7.

Note that since c and gy are both on $(\xi, g\eta)$ and g preserves horospheres centered at ξ,

$$d(gy, c) = |\beta_\xi(gy, c)| + O(\alpha) = |\beta_\xi(gy, a)| + O(\alpha) = |\beta_\xi(o, a)| + O(\alpha) = d(o, a) + O(\alpha). \tag{3.2}$$

By hyperbolicity of the space, there is a uniform constant $O(\alpha)$ which bounds the diameter of this inner triangle. Then by the triangle inequality and (3.2),

$$d(o, gy) \leq d(o, a) + d(a, c) + d(c, gy) \leq 2d(o, a) + O(\alpha).$$

Moreover, let q be the comparable point on a geodesic $[o, g\eta)$ to gy on a geodesic $(g\eta, \xi)$; this means q is the point in the same horosphere centered at $g\eta$ as the point gy. In particular,

$$d(q, b) = |\beta_{g\eta}(q, b)| + O(\alpha) = |\beta_{g\eta}(gy, c)| + O(\alpha) = d(gy, c) + O(\alpha) = d(o, a) + O(\alpha) = d(o, b) + O(\alpha)$$

hence $d(o, q) = 2d(o, a) + O(\alpha)$, and since q is comparable to gy, their distance is bounded above by $O(\alpha)$ by Corollary 2.7. Then we obtain a lower bound

$$2d(o, a) - O(\alpha) \leq d(o, q) - d(gy, q) \leq d(o, gy).$$

Then by the triangle inequality, the fact that g is an isometry, and the upper bound on $d(o, y)$,

$$2d(o, a) - C - O(\alpha) \leq d(o, go) \leq 2d(o, a) + C + O(\alpha). \tag{3.3}$$

Noting that $\langle \xi, g\eta \rangle_o = d(o, a) = \frac{1}{2}d(o, go)$ implies $|\langle \xi, g\eta \rangle_o - \frac{1}{2}d(o, go)| \leq A$ where $A = C + O(\alpha)$. By Definition 2.5, the containments of shadows follow.

Let us now prove the bound on the Busemann functions. By the quasi-cocycle property,

$$\beta_{g\eta}(o, go) = \beta_{g\eta}(o, \xi_t) + \beta_{g\eta}(\xi_t, g\xi_t) + \beta_{g\eta}(g\xi_t, go) + O(\alpha). \tag{3.4}$$

Now, assume that $d(o, go) > 2t$. Then $g\eta \in V(o, \xi, t - A)$, so we have

$$\beta_{g\eta}(o, \xi_t) = t + O(\alpha).$$

Moreover, since the group acts by isometries,

$$\beta_{g\eta}(g\xi_t, go) = \beta_\eta(\xi_t, o).$$

Further, by compactness we can choose a constant D such that K is disjoint from $V(o, \xi, D)$. Hence, by Lemma 2.11 and the antisymmetric and 1-Lipschitz properties of Busemann functions, $\eta \in K$ implies

$$t - 2D - O(\alpha) \leq \beta_\eta(\xi_t, o) \leq t.$$

Finally, since q lies on $[o, g\eta)$,

$$\beta_{g\eta}(o, q) = d(o, q) + O(\alpha)$$

and, as discussed before,

$$d(q, go) \leq d(q, gy) + d(gy, go) \leq O(\alpha) + C$$

hence

$$|\beta_{g\eta}(o, go) - d(o, go)| \leq O(\alpha) + 2C$$

which yields by Equation 3.4 and the preceding equations

$$|\beta_{g\eta}(\xi_t, g\xi_t) - d(o, go) + 2t| \leq B$$

for a suitable choice of B, as required. □

4. Quasi-conformal densities and estimates near the cusps

In this section, we will introduce the background on quasi-conformal densities and prove several key lemmas for the global shadow lemma.

4.1. **Background on quasi-conformal densities.** Let (X, d) be a proper hyperbolic metric space and $\Gamma < \mathrm{Isom}(X, d)$ acting properly discontinuously on X. Then a *quasi-conformal density of dimension* $\delta > 0$ is a family $\{\mu_x\}_{x \in X}$ of mutually absolutely continuous finite non-trivial measures on ∂X with the following properties:

- (quasi-Γ-invariance) there exists $C > 0$ such that for all $\gamma \in \Gamma$, $x \in X$, and a.e. $\eta \in \partial X$, we have

$$C^{-1} \leq \frac{d\gamma_* \mu_x}{d\mu_{\gamma x}}(\eta) \leq C; \tag{4.1}$$

- (transformation rule) there exists $C > 0$ such that, for all $x, y \in X$ and a.e. $\eta \in \partial X$, we have

$$C^{-1} e^{-\delta \beta_\eta(x,y)} \leq \frac{d\mu_x}{d\mu_y}(\eta) \leq C e^{-\delta \beta_\eta(x,y)}. \tag{4.2}$$

If C can be chosen equal to 1, then the density is called a *conformal density*. A measure μ is a δ-(quasi-)conformal measure if $\mu = \mu_x$ for some (quasi-)conformal density $\{\mu_x\}_{x \in X}$ of dimension δ (see in [MYJ20, Proposition 2.5] that this definition agrees with the original definition of quasi-conformal measure, as in [Coo93, Definition 4.1]). Note that any quasi-conformal measure with support contained in Λ_Γ must in fact have support equal to Λ_Γ, because quasi-Γ-invariance and the transformation rule imply that the support is a Γ-invariant set, and Λ_Γ is the smallest closed Γ-invariant set.

A particularly famous example of a conformal density is the *Patterson–Sullivan density*, first constructed by Patterson for Fuchsian groups and extended by Sullivan to geometrically finite actions on hyperbolic spaces ([Pat76, Sul79, Sul84]). We call any density (measure) constructed in such a way a *Patterson–Sullivan density (measure)*. A Patterson–Sullivan density, if it exists, has conformal dimension equal to the critical exponent δ_Γ, which is also the critical exponent of the Poincaré series.

The Patterson–Sullivan construction has been generalized by Coornaert [Coo93] to any non-elementary group Γ of isometries of X when (X, d) is a proper hyperbolic metric space and δ_Γ is finite. Coornaert showed under these assumptions that there exists a Patterson–Sullivan density on Λ_Γ [Coo93, Théorème 5.4]. Coornaert recovers Sullivan's shadow lemma [Sul79] in this setting [Coo93, Proposition 6.1] for a quasi-conformal measure μ of any dimension $\delta > 0$. As a corollary, (1) μ must have conformal dimension at least δ_Γ [Coo93, Corollaire 6.6]; (2) the only points in Λ_Γ that can have positive μ mass are parabolic points; (3) $\delta_\Gamma > 0$ [Coo93, Corollaire 5.5] and (4) the set of parabolic points has full measure if and only if the Poincaré series converges at δ (see e.g. [DOP00, p. 114] for the case of Patterson–Sullivan measure and [MYJ20, Proposition 2.12] in generality). When Γ is geometrically finite, the set of parabolic points is countable [Bow12, Lemma 6.9], hence (4) implies any quasi-conformal density of dimension $\delta > \delta_\Gamma$ has atomic part on the set of parabolic points, since the Poincaré series must converge at δ. On the other hand, (4) implies further that Γ is of divergence type if and only if Patterson–Sullivan measure has no atoms. Hence, [MYJ20, Theorem 4.1, Theorem 5.2] imply that all nonatomic quasi-conformal measures of dimension $\delta > 0$ on Λ_Γ are ergodic and equivalent up to bounded Radon-Nikodym derivative.

The existence of a Patterson–Sullivan measure with no atoms is nontrivial: see for instance the examples of Dal'bo–Otal–Peigné [DOP00, Section 4], which arise from geometrically finite Riemannian manifolds of pinched negative curvature and which have atoms at parabolic points. Patterson–Sullivan density is known to have no atoms in some settings, such as for geometrically finite Riemannian manifolds with pinched negative curvature and parabolic gap ($\delta_\Pi < \delta_\Gamma$ for all parabolic subgroups $\Pi < \Gamma$) [DOP00, Proposition 1], for relatively hyperbolic groups acting on their Cayley graph [Yan21, Proposition 4.1], and for geometrically finite Hilbert geometries (discussed in Section 7.3).

In our hypotheses, we will study quasi-conformal measures on Λ_Γ with no atoms. One appeal of results stated in this generality is that the proof is intrinsic to these defining properties, rather than the Patterson construction.

4.2. **The measure of shadows at parabolic fixed points.** Let us now embark on the proof of our global shadow lemma (Theorem 1.4). Let us remark that, as discussed in Subsection 4.1, Coornaert [Coo93, Proposition 6.1] proved a version of the shadow lemma for shadows that are centered at points on the orbit Γo of a given basepoint: all such points belong to the non-cuspidal part of X. In this paper, we generalize this result by considering shadows centered at *any* point ξ_t, in particular points that may be far from the orbit Γo.

Lemma 4.1. *Let (X, d) be a proper hyperbolic metric space, Γ a group of isometries of X acting properly discontinuously on X and $\{\mu_x\}_{x\in X}$ a quasi-conformal density of dimension δ on Λ_Γ. Fix $o \in C_\Gamma$ and $\xi \in \Lambda_\Gamma$, and let α be the hyperbolicity constant of X. Then for all $\eta \in V(o, \xi, t)$ and $t \geq 0$,*

$$|\beta_\eta(o, \xi_t) - t| \leq O(\alpha)$$

and thus for all $t \geq 0$ and $s \geq -t$,

$$\mu_{\xi_t}(V(o, \xi, t)) \asymp e^{-\delta s}\mu_{\xi_{t+s}}(V(o, \xi, t))$$

with uniform constants, independent of t and s.

Proof. To prove the first part, let p be a closest point projection of η onto $[o, \xi)$. By Lemma 2.2, since $\eta \in V(o, \xi, t)$ we have $d(o, p) \geq t + O(\alpha)$, hence $\beta_p(o, \xi_t) = t + O(\alpha)$ and by Lemma 2.10 we have

$$\beta_\eta(o, \xi_t) = \beta_p(o, \xi_t) + O(\alpha) = t + O(\alpha).$$

The first part implies the second part because, by the transformation rule of conformal densities and the coarse cocycle property of Busemann functions,

$$\begin{aligned}\mu_{\xi_t}(V(o, \xi, t)) &\asymp \int_{V(o,\xi,t)} e^{-\delta\beta_\eta(\xi_t,\xi_{t+s})}\, d\mu_{\xi_{t+s}}(\eta)\\ &\asymp \int_{V(o,\xi,t)} e^{-\delta(-\beta_\eta(o,\xi_t)+\beta_\eta(o,\xi_{t+s}))}\, d\mu_{\xi_{t+s}}(\eta)\\ &\asymp e^{-\delta s}\mu_{\xi_{t+s}}(V(o, \xi, t))\end{aligned}$$

where the cocycle and antisymmetric properties of the Busemann function are applied in the second equality. □

Let Π be a parabolic subgroup of Γ. In the following propositions we will use the following "counting functions": for $t \geq 0$

$$f_\Pi(t) := \sum_{\substack{g\in\Pi \\ d(o,go)\geq 2t}} e^{-\delta d(o,go)+\delta t}$$

and

$$f_\Pi^c(t) := \#\{g \in \Pi \ : \ d(o,go) \leq 2t\} e^{-\delta t}.$$

Lemma 4.2. *Let (X,d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume $\{\mu_x\}_{x\in X}$ a quasi-conformal density of dimension δ on Λ_Γ with no atoms. Let ξ be a bounded parabolic point in Λ_Γ with stabilizer the parabolic subgroup Π, and $o \in X$. Let ξ_t be a point on a geodesic ray $[o,\xi)$ at distance t from o. Then there exist constants A and C depending on ξ and o such that for all $t > A$,*

$$C^{-1}\mu_{\xi_t}(V(o,\xi,t+A)) \leq f_\Pi(t) \leq C\mu_{\xi_t}(V(o,\xi,t-A)),$$

and

$$C^{-1}\mu_{\xi_t}(\partial X \smallsetminus V(o,\xi,t-A)) \leq f_\Pi^c(t) \leq C\mu_{\xi_t}(\partial X \smallsetminus V(o,\xi,t+A)).$$

Proof. First let us show the upper bound. By Lemma 3.7(a), there is a constant A such that for all $t > A$,

$$\bigcup_{\substack{g\in\Pi \\ d(o,go)\geq 2t}} gK \subset V(o,\xi,t-A)$$

where K is a compact fundamental domain for the action of the parabolic subgroup Π on $\Lambda_\Gamma \smallsetminus \{\xi\}$ given by the definition of bounded parabolic. Since Π acts on $\Lambda_\Gamma \smallsetminus \{\xi\}$ properly discontinuously, every point of $\Lambda_\Gamma \smallsetminus \{\xi\}$ is contained in finitely many translates of K, hence for some integer M

$$\sum_{\substack{g\in\Pi \\ d(o,go)\geq 2t}} \mu_{\xi_t}(gK) \leq M\mu_{\xi_t}\left(\bigcup_{\substack{g\in\Pi \\ d(o,go)\geq 2t}} gK\right) \leq M\mu_{\xi_t}(V(o,\xi,t-A)). \tag{4.3}$$

Moreover, Lemma 3.7(a) gives us control over $\beta_{g\eta}(\xi_t, g\xi_t)$ for all such $g \in \Pi$ and all $\eta \in K$, hence (applying the defining properties of a quasi-conformal density),

$$\mu_{\xi_t}(gK) = \int_{gK} \frac{d\mu_{\xi_t}}{d\mu_{g\xi_t}}(\lambda)\ d\mu_{g\xi_t}(\lambda) \asymp \int_{gK} e^{-\delta\beta_\lambda(\xi_t,g\xi_t)}\ d\mu_{g\xi_t}(\lambda) \tag{4.4}$$

and, using Lemma 3.7(a),

$$\asymp \int_{gK} e^{-\delta(d(o,go)-2t)}\ d\mu_{g\xi_t}(\lambda) = e^{-\delta d(o,go)+2\delta t}\mu_{g\xi_t}(gK) \tag{4.5}$$

$$\asymp e^{-\delta d(o,go)+2\delta t}\mu_{\xi_t}(K). \tag{4.6}$$

Since K is compact and disjoint from ξ, there is a constant D such that K is disjoint from $V(o,\xi,D)$, hence by Lemma 2.11, for t sufficiently large and $\eta' \in K$,

$$t - 2D - O(\alpha) = d(o,\xi_t) - 2D - O(\alpha) \leq \beta_{\eta'}(\xi_t, o) \leq t.$$

Then another computation using the defining properties of a conformal density gives

$$\mu_{\xi_t}(K) \asymp e^{-\delta t}\mu_o(K). \tag{4.7}$$

Since K is a fundamental domain for the action of the countable group Π on $\Lambda_\Gamma \setminus \{\xi\}$, and the quasi-conformal densities are absolutely continuous by definition and nonatomic by assumption, $\mu_o(K)$ is some positive constant, so we obtain a constant A' independent of t such that

$$\frac{1}{A'} f_\Pi(t) \le \mu_{\xi_t}(V(o, \xi, t - A)).$$

The argument for the lower bound is similar. By the contrapositive of Lemma 3.7(b), and using that K is a fundamental domain for the parabolic subgroup, there is a constant A such that

$$\big(\Lambda_\Gamma \cap V(o, \xi, t + A)\big) \setminus \{\xi\} \subset \bigcup_{\substack{g \in \Pi \\ d(o,go) \ge 2t}} gK.$$

Then by subadditivity and since the quasi-conformal density is supported on Λ_Γ with no atomic part,

$$\mu_{\xi_t}(V(o, \xi, t + A)) \le \sum_{\substack{g \in \Pi \\ d(o,go) \ge 2t}} \mu_{\xi_t}(gK).$$

Now, by applying the estimates from Equations (4.4) and (4.7), and adjusting the previous constant A' if needed, we have

$$\mu_{\xi_t}(V(o, \xi, t + A)) \le A' f_\Pi(t).$$

The estimate for the complement of the shadow is similar and uses Lemma 3.7 as well, hence the proof is omitted. For more details, see [Sch04, Proposition 3.6]. □

Lemma 4.3. *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume $\{\mu_x\}_{x \in X}$ is a quasi-conformal density of dimension δ on Λ_Γ with no atoms. For any bounded parabolic fixed point ξ whose stabilizer Π has δ-tempered growth, and any $o \in X$, there exists a constant C (note that it depends on all the above) such that for all ξ_t on a geodesic ray $[o, \xi)$ distance t from o,*

$$C^{-1} B_\Pi(2t) e^{-\delta t} \le \mu_{\xi_t}(V(o, \xi, t)) \le C B_\Pi(2t) e^{-\delta t}$$

and

$$C^{-1} B_\Pi(2t) e^{-\delta t} \le \mu_{\xi_t}(\partial X \setminus V(o, \xi, t)) \le C B_\Pi(2t) e^{-\delta t}.$$

Note that, by summing the two equations above, we obtain that the total mass of the measure μ_{ξ_t} grows like $B_\Pi(2t)e^{-\delta t}$. This is possible because ξ_t is far from the orbit Γo.

Proof. Note that we may prove the claim for all t sufficiently large since by adjusting constants, the claim then applies to all $t \ge 0$. Let us write Π as the disjoint union

$$\Pi = \bigcup_{n \in \mathbb{N}} \{g \in \Pi : d(o, go) \in [Rn - R, Rn)\}$$

and denote for any $n \ge 1$

$$a_n := \#\{g \in \Pi \ : \ d(o, go) \in [Rn - R, Rn)\}.$$

First, we claim that, by choosing R large enough, we can make sure that

$$\limsup_{n \to \infty} \frac{1}{R} \log \frac{a_{n+1}}{a_n} < \delta. \tag{4.8}$$

Indeed, since $\delta_\Pi^- > 0$, for any $\epsilon > 0$, there exists an r such that for $R \ge r$, we have $B_\Pi(T - R) \le (1 - \epsilon) B_\Pi(T)$ for all $T > R$. Then

$$\frac{1}{R} \log \frac{a_{n+1}}{a_n} = \frac{1}{R} \log \frac{B_\Pi(nR + R) - B_\Pi(nR)}{B_\Pi(nR) - B_\Pi(nR - R)} \le \frac{1}{R} \log \frac{B_\Pi(nR + R)}{\epsilon B_\Pi(nR)} \le \frac{1}{R} \log \frac{B_\Pi(nR + R)}{B_\Pi(nR)} + \frac{\log(1/\epsilon)}{R}$$

so, since $\delta_\Pi^+ < \delta$, by choosing R large enough, we make sure the right hand side is $< \delta$. Second, we show that

$$\sup_n \sum_{k\geq 0} \frac{a_{n+k}}{a_n} e^{-\delta R k} < \infty.$$

Indeed, from (4.8) there exists $\delta' < \delta R$, $C > 0$ and N such that

$$\frac{a_{n+1}}{a_n} \leq e^{\delta'} \qquad \forall n \geq N \qquad \text{and } \frac{a_{n+1}}{a_n} \leq C \qquad \forall n \leq N.$$

Thus,

$$\sum_{k\geq 0} \frac{a_{n+k}}{a_n} e^{-\delta R k} = \sum_{k\geq 0} \prod_{j=1}^{k} \frac{a_{n+j}}{a_{n+j-1}} e^{-\delta R k} \leq C^N \sum_{k\geq 0} e^{(\delta'-\delta R)k} < \infty$$

as desired. Then for t sufficiently large, a short calculation gives

$$\begin{aligned}\sum_{d(o,go)\geq 2t} e^{-\delta d(o,go)+\delta t} &\asymp e^{\delta t} \sum_{Rn-R\geq 2t} \sum_{\substack{d(o,go)\in\\ [Rn-R,Rn)}} e^{-\delta R n} \\ &= e^{\delta t} \sum_{Rn-R\geq 2t} a_n e^{-\delta R n}\end{aligned}$$

and, setting $n_0 := \lceil \frac{2t}{R} + 1 \rceil$, we have

$$= e^{\delta t} a_{n_0} e^{-\delta R n_0} \sum_{n\geq n_0} \frac{a_n}{a_{n_0}} e^{-\delta R(n-n_0)}$$

and, using that $n_0 R = 2t + O(1)$ and $a_{n_0} \asymp B_\Pi(2t)$,

$$\asymp B_\Pi(2t) e^{-\delta t}$$

thus

$$f_\Pi(t) \asymp B_\Pi(2t) e^{-\delta t}. \tag{4.9}$$

Finally, by Lemma 4.2,

$$\mu_{\xi_t}(V(o,\xi,t-A)) \geq C^{-1} B_\Pi(2t) e^{-\delta t}.$$

An analogous argument for the upper bound gives

$$\mu_{\xi_t}(V(o,\xi,t+A)) \leq C B_\Pi(2t) e^{-\delta t}.$$

Then by the transformation rule and using that $|\beta_\eta(\xi_t, \xi_{t\pm A})| \leq \pm A$ we compare $\mu_{\xi_{t\pm A}}(V(o,\xi,t))$ with $\mu_{\xi_t}(V(o,\xi,t))$ to conclude

$$C^{-1} e^{-\delta A} B_\Pi(2(t+A)) e^{-\delta t} \leq \mu_{\xi_t}(V(o,\xi,t)) \leq C e^{-\delta A} B_\Pi(2(t-A)) e^{-\delta t}$$

and the result follows from the fact that B_Π is nondecreasing.

To estimate the complement of the shadow, the argument is very similar. The calculation of the geometric sum given by Lemma 4.2 immediately yields, by definition,

$$\sum_{d(o,po)\leq 2t} e^{-\delta t} = \#\{g \in \Pi \ : \ d(o,go) \leq 2t\} e^{-\delta t} = B_\Pi(2t) e^{-\delta t}$$

from which the claim follows. ∎

4.3. **Uniform control over all parabolic fixed points.** Note that so far, the constants depend on a particular parabolic point ξ.

Recall by [Bow12, Proposition 6.15] that there are finitely many orbits of parabolic points, hence we express the set of parabolic points $\mathcal{P}$ as a disjoint union of these orbits $\mathcal{P}^1, \dots, \mathcal{P}^a$. For each $i = 1, \dots, a$, pick $p_i \in \mathcal{P}^i$ with stabilizer Π_i, and denote

$$B_i(t) := B_{\Pi_i}(t).$$

Moreover, we choose a quasi-invariant horoball decomposition $\{H_\xi\}_{\xi \in \mathcal{P}}$ of X as given by Proposition 3.3. We now prove a version of the previous lemma where the constants no longer depend on the particular parabolic point chosen.

Lemma 4.4. *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume $\{\mu_x\}_{x \in X}$ is a quasi-conformal density of dimension δ on Λ_Γ with no atoms, and that Γ has δ-tempered parabolic subgroups. Fix a basepoint $o \in X$ and i with $1 \le i \le a$. For $\xi \in \partial X$, let ξ_t denote a point on a geodesic ray $[o, \xi)$ at distance t from o. Then, there exists a constant C such that for all $\xi \in \mathcal{P}^i$ and all times $t > 0$ such that $\xi_t \in H_\xi$, we have*

$$C^{-1} B_i(2d(\xi_t, \Gamma o)) e^{-\delta d(\xi_t, \Gamma o)} \le \mu_{\xi_t}(V(o, \xi, t)) \le C B_i(2d(\xi_t, \Gamma o)) e^{-\delta d(\xi_t, \Gamma o)}$$

and

$$C^{-1} B_i(2d(\xi_t, \Gamma o)) e^{-\delta d(\xi_t, \Gamma o)} \le \mu_{\xi_t}(\partial X \smallsetminus V(o, \xi, t)) \le C B_i(2d(\xi_t, \Gamma o)) e^{-\delta d(\xi_t, \Gamma o)}.$$

Proof. Let $\eta = p_i$, a fixed element of $\mathcal{P}^i$. Let ξ_s be the intersection of $[o, \xi)$ with ∂H_ξ. Similarly, let $\eta_{s'}$ be the intersection of $[o, \eta)$ with ∂H_η. Since the non-cuspidal part is quasi-Γ-invariant (Proposition 3.3), any group element γ for which $\gamma\eta = \xi$ also takes H_η within distance $O(\alpha)$ of H_ξ. Hence, for any such γ, we have that $\gamma\eta_{s'}$ and ξ_s are both within distance $O(\alpha)$ of the ∂H_ξ. Since the parabolic stabilizer of ξ acts cocompactly on $\partial H_\xi \cap C_\Gamma$, and we can choose a fundamental domain with diameter uniformly bounded over all translates of η, we can choose a particular γ such that $\gamma\eta_{s'}$ and ξ_s are uniformly bounded distance apart. Denote this bound by M, and thus

$$d(\xi_s, \gamma o) \le d(\xi_s, \gamma\eta_{s'}) + d(\gamma\eta_{s'}, \gamma o) \le M + s' =: M'. \tag{4.10}$$

Then since geodesic rays meeting at the same boundary point ξ are asymptotic in a hyperbolic metric space (Lemma 2.6),

$$d(\xi_t, \gamma\eta_{t-s}) \le d(\xi_s, \gamma\eta_0) + O(\alpha) = d(\xi_s, \gamma o) + O(\alpha) \le M' + O(\alpha)$$

as well, and by quasi-conformality of the measures (Equation (4.2)) and since Busemann functions are 1-Lipschitz, for any measurable set $E \subset \partial X$,

$$\mu_{\xi_t}(E) \asymp \mu_{\gamma\eta_{t-s}}(E). \tag{4.11}$$

On the other hand, Equation (4.10) suffices to apply Lemma 2.12(2); for all points such as ξ_s and γo which are bounded distance, there is a constant C depending on this bound such that

$$V(\gamma o, \xi, t - s + C) \subset V(\xi_s, \xi, t - s) \subset V(\gamma o, \xi, t - s - C), \tag{4.12}$$

and the containments apply in reverse to the complementary shadow. It follows from the definition that there exists a positive $t_0 = O(\alpha)$ such that for any $t \ge s + t_0$ we have

$$V(\xi_s, \xi, t - s + t_0) \subseteq V(o, \xi, t) \subseteq V(\xi_s, \xi, t - s - t_0).$$

Hence applying Equation (4.11), Equation (4.12), quasi-Γ-invariance of the conformal measures, Γ-equivariance of the shadows, and Lemma 4.3 (it does apply because ξ_t is in H_ξ, so $t - s$ is positive),

$$\begin{aligned} \mu_{\xi_t}(V(o, \xi, t)) &\asymp \mu_{\xi_t}(V(\xi_s, \xi, t - s)) \asymp \mu_{\gamma\eta_{t-s}}(V(\gamma o, \gamma\eta, t - s)) \\ &\asymp \mu_{\eta_{t-s}}(V(o, \eta, t - s)) \asymp B_i(2(t - s)) e^{-\delta(t-s)}, \end{aligned}$$

and again, with similar expressions for the complementary shadow. To conclude the proof, see that $t - s + O(\alpha)$ is the distance of ξ_t to ∂H by Lemma 2.3, which is equal to $d(\xi_t, \Gamma o)$ up to uniform additive constants. $\square$

5. Proof of the global shadow lemma

In this section, we complete the proof of the first main result. Recall the quasi-invariant horoball decomposition $\{H_\xi\}_{\xi \in \mathcal{P}}$ of X as given by Proposition 3.3 and the decomposition $\mathcal{P} = \mathcal{P}^1 \cup \cdots \cup \mathcal{P}^a$ into the finitely many distinct orbits of parabolic points in ∂X. Define the *ith cuspidal part* of X to be

$$X_c^i := \bigcup_{p \in \mathcal{P}^i} H_p \cap C_\Gamma.$$

Recall for each i we choose $p_i \in \mathcal{P}^i$ with stabilizer Π_i, and denote $B_i := B_{\Pi_i}$. Define $b\colon C_\Gamma \to \mathbb{R}$ by

$$b(x) := \begin{cases} 1 & \text{if } x \in X_{nc} \\ B_i(2d(x, \Gamma o)) & \text{if } x \in X_c^i. \end{cases} \tag{5.1}$$

The main result in full generality is:

Theorem 5.1. *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume $\{\mu_x\}_{x \in X}$ is a quasi-conformal density of dimension δ on Λ_Γ with no atoms, and that Γ has δ-tempered parabolic subgroups. Let $o \in C_\Gamma$, and let ξ_t denote the point on a geodesic ray from o to ξ which is distance t from o. Then there exists a constant C such that for any $\xi \in \Lambda_\Gamma$ and any $t > 0$ we have*

$$C^{-1} b(\xi_t) e^{-\delta(t + d(\xi_t, \Gamma o))} \le \mu_o(V(o, \xi, t)) \le C b(\xi_t) e^{-\delta(t + d(\xi_t, \Gamma o))}.$$

Theorem 5.1 is the same as 1.4 from the introduction.

5.1. **Shadows in the non-cuspidal part.** Let us start proving a weak form of the shadow lemma, as in [Coo93, Proposition 6.1].

Lemma 5.2. *Let (X, d) be a hyperbolic metric space and Γ a geometrically finite group of isometries of X. Let $\{\mu_x\}_{x \in X}$ be a quasi-conformal density of dimension δ on Λ_Γ with no atoms. Then for any $t_0 > 0$ there is a constant $C > 0$ such that for all x in X_{nc}, and any $\xi \in \Lambda_\Gamma$,*

$$C^{-1} \le \mu_x(V(x, \xi, t_0)) \le C.$$

Proof. Every point in the non-cuspidal part X_{nc} is uniformly bounded distance from the Γ-orbit of o for any fixed point o in X_{nc}. Let γo be some closest point to x which is in the Γ-orbit of o. Then by quasi-Γ-invariance of the measures and equivariance of shadows,

$$\mu_x(V(x, \xi, t_0)) \asymp \mu_{\gamma o}(V(\gamma o, \xi, t_0)) = \mu_o(V(o, \xi', t_0))$$

where $\xi' = \gamma^{-1}\xi$ varies over Λ_Γ.

First, we claim that there exists $t > 0$ such that for any $\xi, \eta \in \Lambda_\Gamma$, if $\eta \in V(o, \xi, t)$ then $V(o, \xi, t_1) \subseteq V(o, \eta, t_0)$.

Indeed, $\zeta \in V(o, \xi, t)$ implies $\langle \xi, \zeta \rangle_o \ge t$. Moreover, if $\eta \in V(o, \xi, t)$ then $\langle \eta, \xi \rangle_o \ge t$, hence by Equation (2.2) and the fact that the Gromov metric is a metric, one gets

$$c^{-1} e^{-\epsilon \langle \zeta, \eta \rangle_o} \le d_{\partial X}(\zeta, \eta) \le d_{\Lambda_\Gamma}(\zeta, \xi) + d_{\Lambda_\Gamma}(\xi, \eta) \le c e^{-\epsilon \langle \zeta, \xi \rangle_o} + c e^{-\epsilon \langle \xi, \eta \rangle_o} \le 2c e^{-\epsilon t}.$$

Thus, $\langle \zeta, \eta \rangle_o \ge t - \frac{\log(2c^2)}{\epsilon} \ge t_0$ by taking t large enough, which proves the claim.

Then by compactness we can cover Λ_Γ with finitely many shadows of type $V(o, \xi_i, t)$ for $i = 1, \ldots, k$. Now note that, since the support of μ_o is Γ-invariant by quasi-conformality and the action of Γ on Λ_Γ is minimal, then μ_o has full support on Λ_Γ. Then we have

$$C := \inf_i \mu_o(V(o, \xi_i, t)) > 0.$$

Now, let $\xi \in \Lambda_\Gamma$. Then there is a ξ_i such that $\xi \in V(o, \xi_i, t)$, hence by the above claim we have $V(o, \xi_i, t) \subseteq V(o, \xi, t_0)$, so

$$\mu_o(V(o, \xi, t_0)) \geq \mu_o(V(o, \xi_i, t)) \geq C.$$

Now the upper bound is clear, since μ_o is a finite measure. $\square$

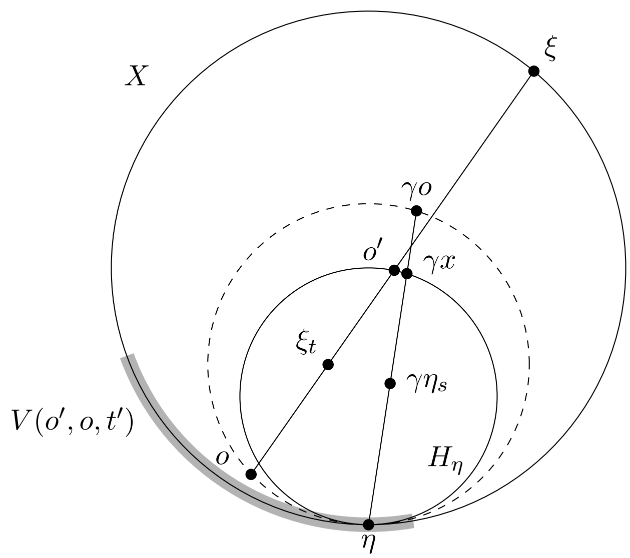

FIGURE 5. Case 2 in the proof of Theorem 5.1.

5.2. **Proof of Theorem 5.1.** With all the ingredients established so far, the proof now follows quite closely the strategy of [Sch04].

Proof of Theorem 5.1. First, by Lemma 4.1 comparing $\mu_{\xi_t}(V(o, \xi, t))$ with $\mu_{\xi_0}(V(o, \xi, t)) = \mu_o(V(o, \xi, t))$, it suffices to show that there is a constant C such that

$$C^{-1} b(\xi_t) e^{-\delta d(\xi_t, \Gamma o)} \leq \mu_{\xi_t}(V(o, \xi, t)) \leq C b(\xi_t) e^{-\delta d(\xi_t, \Gamma o)}. \tag{5.2}$$

The case where ξ_t is in X_{nc} now follows from Lemma 5.2; from the definition of shadows, there exists $t_0 = O(\alpha)$ such that

$$V(o, \xi, t) \supseteq V(\xi_t, \xi, t_0)$$

so Lemma 5.2 applied with $x = \xi_t$ gives the estimate

$$\mu_{\xi_t}(V(o, \xi, t)) \asymp \mu_{\xi_t}(V(\xi_t, \xi, t_0)) \asymp 1.$$

The conclusion follows for ξ_t in X_{nc}, since all such ξ_t are bounded distance from Γo and $b(\xi_t) = 1$.

It remains to consider the case where ξ_t is in the cuspidal part of X. Let i be such that $\xi_t \in X_c^i$. We denote as $\eta \in \mathcal{P}^i$ the boundary point of the horoball to which ξ_t belongs. We have two cases.

Case 1: If $\eta \in V(o, \xi, t)$, then by Lemma 2.12(1)

$$V(o, \eta, t + O(\alpha)) \subseteq V(o, \xi, t) \subseteq V(o, \eta, t - O(\alpha)).$$

By Lemma 2.8,

$$|\beta_\eta(\eta_t,\xi_t)| \le d(\eta_t,\xi_t) \le O(\alpha)$$

hence quasi-conformality yields

$$C^{-1}\mu_{\eta_t}(V(o,\eta,t+O(\alpha))) \le \mu_{\xi_t}(V(o,\xi,t)) \le C\mu_{\eta_t}(V(o,\eta,t-O(\alpha)))$$

where C depends on α and the quasi-conformality constant. The claim follows by Lemma 4.4 and the fact that $d(\eta_t,\Gamma o) = d(\xi_t,\Gamma o) + O(\alpha)$.

Case 2: Suppose that $\eta \notin V(o,\xi,t)$. Let us introduce some notation; see Figure 5 for guidance. Let o' denote the intersection point of a geodesic $[o,\xi)$ with the horosphere ∂H_η centered at η bounding X_{nc}, where o' is closer to ξ than ξ_t. Let $t' = d(\xi_t,o')$. Notice that t' is chosen so that

$$\partial X \smallsetminus V(o',o,t'-O(\alpha)) \subset V(o,\xi,t) \subset \partial X \smallsetminus V(o',o,t'+O(\alpha)).$$

Hence, it suffices to estimate $\mu_{\xi_t}(\partial X \smallsetminus V(o',o,t'))$. Let η_t be the point on a geodesic ray $[o,\eta)$ which is distance t from o. Let γ be an element of the stabilizer of η such that the geodesic ray $\gamma[o,\eta)$ from γo to η intersects the same fundamental domain for the action of the stabilizer of η on ∂H_η as o'. Let $x \in [o,\eta)$ be such that γx is the intersection of the geodesic $\gamma[o,\eta)$ with the horosphere ∂H_η. In particular, the distance between γx and o' is uniformly bounded, independently of η.

The Case 2 assumption implies η is in $V(o',o,t'-O(\alpha))$, and by Lemma 2.12(1) we have

$$V(o',\eta,t'+O(\alpha)) \subset V(o',o,t') \subset V(o',\eta,t'-O(\alpha)).$$

Thus, to estimate $\mu_{\xi_t}(V(o,\xi,t))$, it suffices to estimate

$$\mu_{\xi_t}(\partial X \smallsetminus V(o',\eta,t')).$$

In order to do so, set $s = t'+d(o,x)$. Chose geodesic representatives $[o,o'] \subset [o,\xi)$ and $[x,\eta) \subset [o,\eta)$. Then η is in $V(o',o,t'-O(\alpha))$, so by the fellow traveler property of Lemma 2.8, ξ_t and $\gamma\eta_s$, which are the points at time t' along $[o,o']$ and $\gamma[x,\eta)$ respectively, are uniformly bounded distance apart. Since γx is close to o', we have

$$\mu_{\xi_t}(\partial X \smallsetminus V(o',\eta,t')) \asymp \mu_{\gamma\eta_s}(\partial X \smallsetminus V(\gamma x,\eta,t')) \tag{5.3}$$

and, by shifting perspective along the geodesic, we obtain

$$\asymp \mu_{\gamma\eta_s}(\partial X \smallsetminus V(\gamma o,\eta,s)) \tag{5.4}$$

hence, since $\gamma\eta = \eta$, and by quasi-Γ-invariance,

$$\asymp \mu_{\eta_s}(\partial X \smallsetminus V(o,\eta,s)) \tag{5.5}$$

thus recalling that $\eta \in \mathcal{P}^i$, we have

$$\asymp B_i(2d(\eta_s,\Gamma o))e^{-\delta d(\eta_s,\Gamma o)} \tag{5.6}$$

by direct application of Lemma 4.4. Finally

$$\asymp B_i(2d(\xi_t,\Gamma o))e^{-\delta d(\xi_t,\Gamma o)} \tag{5.7}$$

again because ξ_t is uniformly bounded distance from $\gamma\eta_s$. Recalling that $b(\xi_t) = B_i(2d(\xi_t,\Gamma o))$ yields (5.2), thus completing the proof. □

5.3. **Corollaries of the global shadow lemma.** We now prove Corollaries 1.3 and 1.7 from the introduction.

Proof of Corollary 1.3. By Equation (2.2), there exists $A > 0$ such that

$$V(o, \xi, \epsilon^{-1} \log(r^{-1}) + A) \subset D(\xi, r) \subset D(\xi, 2r) \subset V(o, \xi, \epsilon^{-1} \log(r^{-1}) - A)$$

for any $\xi \in \Lambda_\Gamma$, any $r > 0$. By the triangle inequality, for any $t \geq A$

$$|d(\xi_{t-A}, \Gamma o) - d(\xi_{t+A}, \Gamma o)| \leq d(\xi_{t-A}, \xi_{t+A}) = 2A.$$

Then setting $t = \epsilon^{-1} \log(r^{-1})$,

$$1 \leq \frac{\mu_o(D(\xi, 2r))}{\mu_o(D(\xi, r))} \leq C^2 e^{\delta 4A} \frac{b(\xi_{t-A})}{b(\xi_{t+A})}.$$

If ξ_{t-A} and ξ_{t+A} are in a horoball H_ξ centered at the same parabolic point ξ in $\mathcal{P}^i$, then by Remark 3.5(3), there is a constant C' such that for all $t \geq A$,

$$\frac{b(\xi_{t-A})}{b(\xi_{t+A})} = \frac{B_i(2d(\xi_{t-A}, \Gamma o))}{B_i(2d(\xi_{t+A}, \Gamma o))} \leq C'$$

which yields the estimate. Else, $t \leq A$ hence $d(\xi_{t+A}, \Gamma o) \leq d(o, \Gamma o) + 2A$ is uniformly bounded above, so there exist $C'' > 0$ independent of ξ such that $\mu_o(D(\xi, r)) \geq \mu_o(V(o, \xi, t + A)) \geq C''$, which completes the proof, since μ_o is a finite measure. □

Proof of Corollary 1.7. Suppose by contradiction that the harmonic measure ν and the Patterson–Sullivan measure μ are in the same measure class. By [GT20, Proposition 5.1], the Radon-Nykodim derivative $\frac{d\mu}{d\nu}$ is bounded away from 0 and infinity. Now, for any $g \in \Gamma$ let $\xi \in \Lambda_\Gamma$ such that go lies within distance $O(\alpha)$ of a geodesic ray $[o, \xi)$. By the shadow lemma for the hitting measure ([GT20, Proposition 2.3]), we have

$$\nu(V(o, \xi, d(o, go))) \asymp e^{-d_G(e,g)}$$

where d_G is the Green distance (see e.g. [GT20, Section 2.5]). On the other hand, by Theorem 1.4,

$$\mu(V(o, \xi, d(o, go))) \asymp e^{-\delta_\Gamma d(o,go)}$$

so, since $\frac{d\mu}{d\nu}$ is bounded above and below, the difference $d_G(e, g) - \delta_\Gamma d(o, go)$ is bounded independently of g. Since the Green metric d_G is quasi-isometric to any word metric on Γ, this implies that the orbit map $g \mapsto go$ is a quasi-isometric embedding; however, by letting $g = h^n$ with h a parabolic element and taking the limit as $n \to \infty$, we obtain a contradiction. □

6. Applications of the shadow lemma

Recall that if p is a boundary point, then $H_p(r)$ is the unique horoball centered at p with radius r and $\mathcal{H}_p(r)$ is the shadow in ∂X of $H_p(r)$. Recall that $\mathcal{P}$ is the set of all parabolic fixed points in ∂X, which we decompose as a disjoint union of orbits $\mathcal{P} = \mathcal{P}^1 \cup \cdots \cup \mathcal{P}^a$. Then we fix a quasi-$\Gamma$-invariant horoball decomposition of X as given by Proposition 3.3, where each parabolic point p in the set $\mathcal{P}$ determines a unique horoball H_p centered at p in the decomposition, and we denote the radius of H_p by r_p.

6.1. **Dirichlet Theorem.** We now prove the Dirichlet-type theorem, which does not rely on the shadow lemma. For fixed $s > 0$, recall the set of parabolic points with large radius is denoted by

$$\mathcal{P}_s := \{p \in \mathcal{P} \mid r_p \geq s\}.$$

We also denote the set of parabolic points in a given orbit with large radius by

$$\mathcal{P}_s^i := \{p \in \mathcal{P}^i \mid r_p \geq s\}.$$

Theorem 6.1 (Dirichlet-type theorem)**.** *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X with parabolic elements. Then there exist constants $c_1 > 0, c_2 \geq 1$ such that for all $s \leq c_1$, the set*

$$\bigcup_{p \in \mathcal{P}_s} \mathcal{H}_p(c_2\sqrt{sr_p})$$

covers the limit set Λ_Γ, and there exists $0 < c_3 \leq 1$ such that the shadows $\{\mathcal{H}_p(c_3\sqrt{sr_p})\}_{p\in\mathcal{P}_s}$ are pairwise disjoint.

Note that Theorem 6.1 is effectively the same statement as Theorem 1.5.

Proof. First, by cocompactness of the action of Γ on the non-cuspidal part, note that we can rescale all horoballs in each of the finitely many Γ-orbits of parabolic points by a multiplicative constant c so that the convex hull of the limit set C_Γ is covered by the horoballs rescaled by c, i.e.

$$C_\Gamma \subseteq \bigcup_{p\in\mathcal{P}} H_p(cr_p). \tag{6.1}$$

Now fix $0 < s \leq c_1 := \frac{1}{c}$ and $\xi \in \Lambda_\Gamma$. Let $w \in [o, \xi)$ such that $e^{-d(o,w)} = cs$. By the above Equation (6.1), there is some $p \in \mathcal{P}$ such that $w \in H = H_p(cr_p)$. Let q be a point on the intersection of $[o, p)$ with ∂H, so that $cr_p = e^{-\beta_p(o,q)}$ by the definition of radius of a horoball. Since $w \in H$, we have $\beta_p(o, w) \geq \beta_p(o, q)$. Since $w \in [o, \xi)$, we apply Lemma 2.15 and conclude there exists a point z on $[o, \xi)$ with

$$\beta_p(o, z) \geq \frac{d(o, q) + d(o, w)}{2} - O(\alpha).$$

Then there exists a constant c_2 such that

$$e^{-\beta_p(o,z)} \leq e^{-\frac{d(o,q)+d(o,w)}{2}} e^{O(\alpha)} = c_2\sqrt{r_p s}$$

which shows that z belongs to $H_p(c_2\sqrt{r_p s})$, hence also ξ belongs to $\mathcal{H}_p(c_2\sqrt{r_p s})$. Finally, observe that $s \leq r_p$ since

$$cs = e^{-d(o,w)} \leq e^{-\beta_p(o,w)} \leq e^{-\beta_p(o,q)} = cr_p.$$

To prove the second part, note that, since the horoballs are disjoint, we have by Corollary 2.17 that there exists a constant $C > 0$ for which

$$d_{\partial X}(p_1, p_2) \geq C(r_1 r_2)^{\frac{\epsilon}{2}}.$$

Now, by Lemma 2.13, there exists a constant c_3 such that for each $i = 1, 2$ one has the following bound on the diameter of the shadow

$$\operatorname{diam} \mathcal{H}_{p_i}(c_3\sqrt{r_i s}) \leq \frac{C}{4}(r_i s)^{\frac{\epsilon}{2}}.$$

Hence, using that $r_i \geq s$, the inequalities

$$d_{\partial X}(p_1, p_2) \geq C(r_1 r_2)^{\frac{\epsilon}{2}} \geq \frac{C}{2}(sr_1)^{\frac{\epsilon}{2}} + \frac{C}{2}(sr_2)^{\frac{\epsilon}{2}} > \operatorname{diam} \mathcal{H}_{p_1}(c_3\sqrt{r_1 s}) + \operatorname{diam} \mathcal{H}_{p_2}(c_3\sqrt{r_2 s})$$

show that the shadows $\mathcal{H}_{p_1}(c_3\sqrt{r_1 s})$ and $\mathcal{H}_{p_2}(c_3\sqrt{r_2 s})$ are disjoint. □

6.2. **Horoball counting.** Now we will apply the Dirichlet theorem to produce horoball counting estimates. We need a version of the shadow lemma for shadows of horoballs rather than traditional shadows. The following condition will be the main hypothesis on the measures for the remaining application, so we introduce it as a definition.

Definition 6.2 (Horoball shadow lemma)**.** Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. We say that a measure μ on ∂X *satisfies the horoball shadow lemma with dimension* δ if for all $c_1 < c_2$ there there exists a multiplicative constant such that

$$\mu(\mathcal{H}_p(c\theta r_p)) \asymp B_i(-2\log\theta)\theta^{2\delta} r_p^{\delta}. \tag{6.2}$$

for any $0 < \theta \leq 1$, any $c \in [c_1, c_2]$, any $p \in \mathcal{P}^i$, and any $i = 1, \dots, a$.

The measures we have considered so far satisfy this property:

Corollary 6.3. *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X with δ-tempered parabolic subgroups. Then any δ-quasi-conformal measure μ on Λ_Γ with no atoms satisfies the horoball shadow lemma with dimension δ.*

Proof. Letting $t = -\log(c\theta r_p)$, see that by definition of r_p, we have $-\log(\theta) - k \leq d(\xi_t, \Gamma o) \leq -\log(\theta) + k$ for some constant k depending only on α and the fixed interval containing c. Then Equation 6.2 follows from the global shadow lemma (Theorem 5.1), Lemma 2.14, and Remark 3.5(1) and (3). □

The following horoball counting statement is analogous to [SV95, Theorem 3].

Proposition 6.4 (Horoball counting)**.** *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume Γ has δ-tempered parabolic subgroups, and μ is a measure on Λ_Γ that satisfies the horoball shadow lemma with dimension δ (Definition 6.2). Let us define*

$$\mathcal{P}_n(\lambda) := \{p \in \mathcal{P},\ \lambda^{n+1} \leq r_p \leq \lambda^n\}.$$

Then there exist $\lambda < 1$ and constants such that

$$\#\mathcal{P}_n(\lambda) \asymp \lambda^{-n\delta}$$

for all $n \in \mathbb{N}$.

Proof. Let c_1, c_2 and c_3 be as in Theorem 6.1. Then for all $0 < s < c_1$,

$$\mu(\Lambda_\Gamma) \leq \sum_{p \in \mathcal{P}_s} \mu(\mathcal{H}_p(c_2\sqrt{s r_p}))$$

and

$$\sum_{p \in \mathcal{P}_s} \mu(\mathcal{H}_p(c_3\sqrt{s r_p})) \leq \mu(\Lambda_\Gamma).$$

Then applying the horoball shadow lemma (Definition 6.2) with $\theta = \sqrt{s/r_p}$ to get

$$\sum_{i=1}^{a} \sum_{p \in \mathcal{P}_s^i} B_i\left(-\log\left(\frac{s}{r_p}\right)\right)\left(\frac{s}{r_p}\right)^{\delta} r_p^{\delta} \asymp 1$$

hence there is a constant $c > 0$ such that

$$c^{-1}s^{-\delta} \leq \sum_{i=1}^{a} \sum_{p \in \mathcal{P}_s^i} B_i\left(\log\frac{r_p}{s}\right) \leq c s^{-\delta}. \tag{6.3}$$

Now by finiteness of the upper annular growth rate, for u sufficiently large, $t \geq \log(u)$, and $i = 1, \dots, a$,

$$\frac{B_i(t)}{B_i(t - \log u)} \leq u^{\delta^+_{\Pi_i}}$$

where Π_i is the stabilizer of a fixed $p_i \in \mathcal{P}$.

Then since Γ has δ-tempered parabolic subgroups, we may fix a sufficiently large u such that

$$\sum_{i=1}^{a} \sum_{\substack{p \in \mathcal{P}^i \\ su \leq r_p}} B_i(\log(\frac{r_p}{s})) = \sum_{i=1}^{a} \sum_{\substack{p \in \mathcal{P}^i \\ su \leq r_p}} B_i(\log(\frac{r_p}{su}))\frac{B_i(\log(\frac{r_p}{s}))}{B_i(\log(\frac{r_p}{su}))} \leq c(su)^{-\delta} u^{\max_i \delta^+_{\Pi_i}} \leq \frac{1}{2}c^{-1}s^{-\delta}$$

where c is given by Equation (6.3). Then

$$\begin{aligned} \#\{p \in \mathcal{P} : s \leq r_p \leq su\} &\asymp \sum_{i=1}^{a} \sum_{\substack{p \in \mathcal{P}^i \\ s \leq r_p < su}} B_i(\log(\frac{r_p}{s})) \\ &= \sum_{i=1}^{a} \sum_{\substack{p \in \mathcal{P}^i \\ s \leq r_p}} B_i(\log(\frac{r_p}{s})) - \sum_{i=1}^{a} \sum_{\substack{p \in \mathcal{P}^i \\ su \leq r_p}} B_i(\log(\frac{r_p}{s})) \\ &\geq c^{-1}s^{-\delta} - \frac{1}{2}c^{-1}s^{-\delta} \geq \frac{1}{2}c^{-1}s^{-\delta}. \end{aligned}$$

Equation (6.3) and nonnegativity of B_i implies the same expression is bounded above by $cs^{-\delta}$. Taking $\lambda = u^{-1}$ and $s = \lambda^{n+1}$ proves the statement. □

Remark 6.5. We note that by [Yan19, Theorem 1.7], exponential growth of horoballs in Proposition 6.4 is equivalent to the Dal'bo–Otal–Peigné (DOP) condition. Hence, we see that tempered growth implies the DOP condition.

Proposition 6.6 (Horoball counting for distinct orbits)**.** *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume Γ has δ-tempered parabolic subgroups, and μ is a measure on Λ_Γ that satisfies the horoball shadow lemma with dimension δ (Definition 6.2). For each $i = 1, \dots, a$, let us define*

$$\mathcal{P}^i_n(\lambda) := \{p \in \mathcal{P}^i,\ \lambda^{n+1} \leq r_p \leq \lambda^n\}.$$

Then there exists a multiplicative constant such that for $\lambda < 1$ sufficiently small,

$$\#\mathcal{P}^i_n(\lambda) \asymp \lambda^{-n\delta}$$

for all $n \in \mathbb{N}$ and all $i = 1, \dots, a$.

Proof. For each $i = 1, \dots, a$, choose a parabolic point $p_i \in \mathcal{P}^i$, and let $\Pi_i := \mathrm{Stab}(p_i)$ be its stabilizer. For each $t > 0$, consider the function

$$f_i(t) := \#\{g\Pi_i \in \Gamma/\Pi_i \ : \ d(o, g\Pi_i o) \leq t\}.$$

Then, by [HP04, Theorem 3.1], for any i, j and any $t > 0$ we have

$$f_i(t) \asymp \#\{g \in \Gamma \ : \ d(o, go) \leq t\} \asymp f_j(t). \tag{6.4}$$

Now, note that, there exists a constant C, depending on α and the diameter of a fundamental domain for the action of Π_i on the horosphere containing $\Pi_i o$, such that for any $g \in \Gamma$,

$$|d(o, g\Pi_i o) + \log(r_{gp_i})| \leq C$$

hence, by definition of $\mathcal{P}^i_n(\lambda)$ and Equation (6.4), for any i, j and any $n \geq 0$,

$$\#\mathcal{P}^i_n(\lambda) \asymp f_i(n \log(1/\lambda)) \asymp f_j(n \log(1/\lambda)) \asymp \#\mathcal{P}^j_n(\lambda).$$

Since $\#\mathcal{P}_n(\lambda)$ is the sum of the finitely many $\#\mathcal{P}_n^i(\lambda)$, we obtain by Proposition 6.4

$$\#\mathcal{P}_n^i(\lambda) \asymp \#\mathcal{P}_n(\lambda) \asymp \lambda^{-n\delta}.$$

□

6.3. **Khinchin functions.** A *Khinchin function* is a positive, increasing function $\varphi : \mathbb{R}^+ \to (0,1]$ such that there exist constants $b_1 > 1, b_2 > 0$ for which

$$\varphi(b_1 x) \geq b_2 \varphi(x) \qquad \text{for any } x \in \mathbb{R}^+.$$

Note that it follows that for any $k_1 > b_1$ there exists a k_2 such that $\varphi(k_1 x) \geq k_2 \varphi(x)$ for all $x \in \mathbb{R}^+$.

Khinchin functions have been introduced in diophantine approximation: Khinchin's classical theorem [Khi26] states that the set of reals x such that $|x - \frac{p}{q}| < \frac{\psi(q)}{q}$ for infinitely many rationals $\frac{p}{q}$ has measure zero if and only if $\sum_{q=1}^{\infty} \psi(q) < \infty$, and full measure otherwise. The function φ we are using in this paper is related to ψ by the formula $\varphi(x) = \psi(\log x^{-1})$. As a famous example, setting $\varphi(x) = (\log x^{-1})^{-(1+\epsilon)}$ and using Khinchin's original theorem one proves that the set $\{x \in \mathbb{R} \ : \ \exists \text{ infinitely many } \frac{p}{q} \text{ with } |x - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}}\}$ has zero Lebesgue measure if $\epsilon > 0$, and full measure if $\epsilon = 0$.

Now fix (X, d) a hyperbolic metric space, and Γ a geometrically finite group of isometries of X. Let μ be a quasi-conformal measure on Λ_Γ with no atoms. Recall that $H_p(r)$ is the unique horoball centered at the boundary point p with radius r. Note that for any measure μ which satisfies the horoball shadow lemma (Definition 6.2) and for any Khinchin function φ,

$$\mu(\mathcal{H}_p(r_p \varphi(r_p))) \asymp r_p^{\delta} (\varphi(r_p))^{2\delta} B_i(-2 \log \varphi(r_p)) \tag{6.5}$$

where p belongs to $\mathcal{P}^i$.

6.4. **Quasi-independence.** For $i = 1, \ldots, a$, let S_n^i be the union of the shadows of $H_p(r_p \varphi(r_p))$ for $\lambda^n \leq r_p \leq \lambda^{n+1}$ and $p \in \mathcal{P}^i$. Note that S_n^i depends on λ and φ.

Given a horoball H of radius r and a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, we denote as fH the horoball with the same boundary point as H and radius $rf(r)$.

Lemma 6.7 (Quasi-independence). *Let (X, d) be a hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume Γ has δ-tempered parabolic subgroups, and μ is a measure on Λ_Γ that satisfies the horoball shadow lemma with dimension δ (Definition 6.2). Fix a Khinchin function φ. Then there exists a positive constant C such that for all i, j, for all $n, m \in \mathbb{N}$ sufficiently large, and for all $\lambda < 1$ sufficiently small,*

$$\mu(S_n^i \cap S_m^j) \leq C \mu(S_n^i) \mu(S_m^j).$$

Proof. Let $\lambda < 1$ be sufficiently small as given by Proposition 6.6. We denote as $S(H)$ the shadow of the horoball H.

Let $r_i = r_{p_i}$ for $i = 1, 2$. Let $H_{p_1} = H_{p_1}(r_1)$ and $H_{p_2} = H_{p_2}(r_2)$ be two disjoint horoballs. By Corollary 2.17, we obtain

$$d_{\partial X}(p_1, p_2) \geq C_\alpha (r_1 r_2)^{\frac{\epsilon}{2}}.$$

where $C_\alpha > 0$ only depends on the hyperbolicity constant.

Claim. There exists a constant c such that, if $r_1 > r_2$ and $S(\varphi H_{p_1}) \cap S(\varphi H_{p_2}) \neq \emptyset$, then

$$S(H_{p_2}) \subseteq S(c \varphi H_{p_1}). \tag{6.6}$$

Proof of the claim. Since φ is increasing,

$$d_{\partial X}(p_1, p_2) \leq C(\varphi(r_1) r_1)^{\epsilon} + C(\varphi(r_2) r_2)^{\epsilon} \leq 2C(\varphi(r_1) r_1)^{\epsilon}$$

where C comes from Lemma 2.13, hence

$$C_\alpha(r_1 r_2)^{\frac{\epsilon}{2}} \leq 2C(\varphi(r_1)r_1)^\epsilon$$

thus since φ is increasing and r_1 is bounded, φ and hence φ^ϵ is bounded by some constant M and

$$\frac{M(\varphi(r_1)r_1)^\epsilon}{r_2^\epsilon} \geq \left(\frac{\varphi(r_1)^2 r_1}{r_2}\right)^\epsilon \geq \frac{C_\alpha^2}{4C^2}.$$

Hence, if $\xi \in \mathcal{H}_{p_2}(r_2)$, we estimate

$$d_{\partial X}(\xi, p_1) \leq d_{\partial X}(\xi, p_2) + d_{\partial X}(p_2, p_1) \leq C r_2^\epsilon + 2C(\varphi(r_1)r_1)^\epsilon \leq c(\varphi(r_1)r_1)^\epsilon$$

with $c = \frac{M4C^3}{C_\alpha^2} + 2C$, which implies Equation (6.6). $\square$

Now let $m > n$, and pick an element $p_\star$ of $\mathcal{P}_n^i(\lambda)$. Let us consider the set

$$I(p_\star) := \{p \in \mathcal{P}_m^j(\lambda) \;:\; S(\varphi H_p) \cap S(\varphi H_{p_\star}) \neq \emptyset\}.$$

By the horoball shadow lemma (Definition 6.2), for any $p \in \mathcal{P}_m^j(\lambda)$ we have

$$\mu(S(H_p)) \asymp \lambda^{m\delta} \tag{6.7}$$

while by the counting lemma (Prop. 6.4)

$$\#\mathcal{P}_m^j(\lambda) \asymp \lambda^{-m\delta}. \tag{6.8}$$

By Theorem 6.1, setting $s = \lambda^{m+1}$, there exists c_2 such that the shadows in the set

$$\Sigma := \{S(c_2 H_p) \;:\; p \in \mathcal{P}_m^j(\lambda)\}$$

are mutually disjoint. Since $\varphi(x) \leq 1$, we also have that the shadows in the set

$$\Sigma_\varphi := \{S(c_2 \varphi H_p) \;:\; p \in \mathcal{P}_m^j(\lambda)\}$$

are mutually disjoint. Now, by the horoball shadow lemma (Definition 6.2), we have

$$\mu(S(c_2 H_p)) \asymp \mu(S(H_p)), \qquad \mu(S(c_2 \varphi H_p)) \asymp \mu(S(\varphi H_p))$$

for any $p \in \mathcal{P}_m^j(\lambda)$. Hence, since the elements of Σ_φ are pairwise disjoint, applying Equations (6.7) and (6.8) we obtain, for any $p \in \mathcal{P}_m^j(\lambda)$,

$$\mu(S_m^j) \asymp \#\mathcal{P}_m^j(\lambda) \cdot \mu(S(\varphi H_p)) \asymp \frac{\mu(S(\varphi H_p))}{\mu(S(H_p))}. \tag{6.9}$$

Note that the same argument implies $\mu(S_n^i) \asymp \sum_{p \in \mathcal{P}_n^i(\lambda)} \mu(S(\varphi H_p))$, even if the union is not disjoint. Now, note that, if $p \in I(p_\star)$, then by Equation (6.6)

$$S(H_p) \subseteq S(c\varphi H_{p_\star}).$$

Moreover, since the elements of Σ are disjoint and $\mu(S(c_2 H_p)) \asymp \mu(S(H_p))$,

$$\mu(S(\varphi H_{p_\star})) \asymp \mu(S(c\varphi H_{p_\star})) \gtrsim \#I(p_\star) \inf_{p \in I(p_\star)} \mu(S(H_p))$$

hence

$$\begin{aligned}\mu(S^i_n \cap S^j_m) &\le \sum_{p_\star \in \mathcal{P}^i_n(\lambda)} \sum_{p \in I(p_\star)} \mu(S(\varphi H_p)) \\ &\le \sum_{p_\star \in \mathcal{P}^i_n(\lambda)} \# I(p_\star) \sup_{p \in I(p_\star)} \mu(S(\varphi H_p)) \\ &\lesssim \sum_{p_\star \in \mathcal{P}^i_n(\lambda)} \frac{\mu(S(\varphi H_{p_\star}))}{\inf_{p \in I(p_\star)} \mu(S(H_p))} \sup_{p \in I(p_\star)} \mu(S(\varphi H_p)) \\ &\lesssim \mu(S^i_n) \frac{\sup_{p \in I(p_\star)} \mu(S(\varphi H_p))}{\inf_{p \in I(p_\star)} \mu(S(H_p))} \asymp \mu(S^i_n)\mu(S^j_m)\end{aligned}$$

where the last comparison follows by Equation (6.9). This completes the proof. □

6.5. **Khinchin theorem.** Given a Khinchin function φ, a small enough $\lambda < 1$, and $i = 1, \dots, a$, we define the set

$$\Theta^i_\lambda(\varphi) := \limsup_{n \to \infty} S^i_n = \bigcap_{n=0}^{\infty} \bigcup_{m \ge n} \bigcup_{p \in \mathcal{P}^i_m(\lambda)} \mathcal{H}_p(r_p \varphi(r_p)).$$

Moreover, we have the *Khinchin series*

$$K^i_\lambda(\varphi) := \sum_{n=0}^{\infty} \varphi(\lambda^n)^{2\delta} B_i(-2\log \varphi(\lambda^n)).$$

Similarly, we define

$$\Theta_\lambda(\varphi) := \bigcup_{i=1}^{a} \Theta^i_\lambda(\varphi) \qquad \text{and} \qquad K_\lambda(\varphi) := \sum_{i=1}^{a} K^i_\lambda(\varphi).$$

We are now ready to state the main theorem of this subsection.

Theorem 6.8 (Khinchin-type theorem)**.** *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume Γ has δ-tempered parabolic subgroups, and μ is a quasi-conformal probability measure of dimension δ on Λ_Γ with no atoms. Let φ be a Khinchin function. Then there exists a $\lambda < 1$ such that for each $i = 1, \dots, a$:*

(1) $\mu(\Theta^i_\lambda(\varphi)) = 0$ if $K^i_\lambda(\varphi) < \infty$;
(2) $\mu(\Theta^i_\lambda(\varphi)) = 1$ if $K^i_\lambda(\varphi) = \infty$.

As a consequence, $\mu(\Theta_\lambda(\varphi)) = 0$ if and only if $K_\lambda(\varphi) < \infty$, and otherwise $\mu(\Theta_\lambda(\varphi)) = 1$.

To prove this, let us recall the Borel-Cantelli lemma and its converse (for a proof see e.g. [Lam63]):

Lemma 6.9. *Let $(S, \mathbb{P})$ be a probability measure space, and $(A_n) \subseteq S$ a sequence of measurable subsets. Then:*

(1) If $\sum_{n=0}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$.
(2) If $\sum_{n=0}^{\infty} \mathbb{P}(A_n) = \infty$ and there exists $c > 0$ such that $\mathbb{P}(A_n \cap A_m) \le c\mathbb{P}(A_n)\mathbb{P}(A_m)$ for any distinct $n, m \ge 0$, then $\mathbb{P}(\limsup A_n) > 0$.

Proof of Theorem 6.8. Fix $\lambda < 1$ small as given by Proposition 6.6. Note that by Corollary 6.3 and Equation (6.5), for any $p \in \mathcal{P}^i_n(\lambda)$,

$$\mu(S^i_n) \asymp \#\mathcal{P}^i_n(\lambda) \cdot \mu(S(\varphi H_p)) \asymp \lambda^{-n\delta} \lambda^{n\delta} \varphi(\lambda^n)^{2\delta} B_i(-2\log\varphi(\lambda^n)) = \varphi(\lambda^n)^{2\delta} B_i(-2\log\varphi(\lambda^n)).$$

Now, (1) follows from Lemma 6.9 (1).

Conversely, (2) follows Lemma 6.9 (2): using the quasi-independence from Lemma 6.7 we obtain that $\mu(\Theta^i_\lambda(\varphi)) > 0$. Moreover, from [Str94, Lemma 1.2.3] we have that $\Theta^i_\lambda(\varphi)$ is Γ-invariant up to measure zero, meaning that for any $g \in \Gamma$ we have $\mu(g\Theta^i_\lambda(\varphi)\Delta\Theta^i_\lambda(\varphi)) = 0$ (their proof is stated in the convex cocompact case, but the same proof applies here). Thus, from ergodicity of nonatomic quasi-conformal densities [MYJ20, Theorem 4.1] we conclude that $\mu(\Theta^i_\lambda(\varphi)) = 1$. □

6.6. **The logarithm law.** We now state and prove the logarithm law in the general case of hyperbolic metric spaces. The following result compares to [SV95, Proposition 4.9].

Theorem 6.10 (Logarithm Law)**.** *Let (X, d) be a proper hyperbolic metric space and Γ a geometrically finite group of isometries of X. Assume Γ has δ-tempered parabolic subgroups, and μ is a quasi-confomal measure of dimension δ on Λ_Γ with no atoms. If the parabolic subgroups of Γ moreover have mixed exponential growth, then for μ-almost every ξ in the limit set Λ_Γ,*

$$\limsup_{t\to+\infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \frac{1}{2(\delta - \delta_{\max})}$$

where $\delta_{\max}$ is the maximal growth rate of any parabolic subgroup, and ξ_t is the point on a geodesic ray $[o, \xi)$ that is distance t from o.

We will see in Section 7.6 how Theorem 6.10 implies Theorem 1.6.

Proof. We recall the set-up for the proof provided in Stratmann-Velani [SV95]. For any ϵ we define for $0 \le x \le e^{-1}$

$$\varphi_\epsilon(x) := (\log x^{-1})^{-\frac{1+\epsilon}{2(\delta-\delta_{\max})}}$$

where $\delta_{\max} := \max\{\delta_{\Pi_i}, 1 \le i \le a\} < \delta$, and for $x \ge e^{-1}$ we let $\varphi_\epsilon(x) := 1$. Observe that φ_ϵ is a Khinchin function and that φ_ϵ, hence $\Theta_\lambda(\varphi_\epsilon)$, is decreasing in ϵ.

Now, we claim that the Khinchin series $K(\varphi_\epsilon)$ converges if $\epsilon > 0$ and diverges if $\epsilon = 0$. To see this, recall that mixed exponential growth implies that for any i there exist $\delta_i = \delta_{\Pi_i}$ and $a_i \ge 0$ such that $B_i(t) \asymp e^{\delta_i t}(t + 1)^{a_i}$, so we compute

$$\begin{aligned} K^i_\lambda(\varphi_\epsilon) &= \sum_{n=1}^{\infty} \varphi_\epsilon(\lambda^n)^{2\delta} B_i(-2\log\varphi_\epsilon(\lambda^n)) \\ &\asymp \sum_{n=1}^{\infty} (n\log\lambda^{-1})^{-\frac{(1+\epsilon)(\delta-\delta_i)}{\delta-\delta_{\max}}} \left(\frac{1+\epsilon}{\delta-\delta_{\max}}\log(n\log\lambda^{-1}) + 1\right)^{a_i} \\ &\asymp \sum_{n=1}^{\infty} n^{-(1+\epsilon)\frac{\delta-\delta_i}{\delta-\delta_{\max}}} \log(n+1)^{a_i}. \end{aligned}$$

Now, if $\epsilon > 0$ the above series converges for any $i > 0$, while if $\epsilon = 0$ it diverges if $\delta_i = \delta_{\max}$.

Then by Theorem 6.8, the limsup set $\Theta_\lambda(\varphi_\epsilon)$ with respect to φ_ϵ is μ-null for all $\epsilon > 0$, and has full measure for $\epsilon = 0$. Choose a boundary point ξ in the full measure set $\Theta_\lambda(\varphi_0) \setminus \bigcup_{\epsilon>0} \Theta_\lambda(\varphi_\epsilon)$ and a geodesic ray $[o, \xi)$.

Define for each $p \in \mathcal{P}$ the enlarged horoball $\tilde{H}_p := H_p(cr_p)$, where $c = O(\alpha)$ is chosen so that, if a geodesic ray from o to $\xi \in \partial X$ intersects H_p, then *any* geodesic ray from o to ξ intersects $\tilde{H}_p$.

By definition of the limsup sets, there exists a sequence of parabolic points p_n in $\mathcal{P}$ with $r_{p_n} \le 1$ such that $[o, \xi)$ passes through horoballs $\varphi_0\tilde{H}_{p_n}$ in order, and passes through no other horoballs of the form $\varphi_0 H_p$; in other words, the radii $r_{p_n}\varphi_0(r_{p_n})$ are monotone decreasing in n, and $[o, \xi) \cap \varphi_0 H_p \ne \varnothing$ implies $p = p_n$ for some n.

For each n, choose ϵ_n so that the geodesic $[o, \xi)$ is tangent to the horoball $\varphi_{\epsilon_n} H_{p_n}$. More precisely, let ξ_{t_n} be a closest point projection of p_n onto $[o, \xi)$, and choose ϵ_n such that the boundary of $\varphi_{\epsilon_n} H_{p_n}$ contains ξ_{t_n}. See Figure 6 for an illustration.

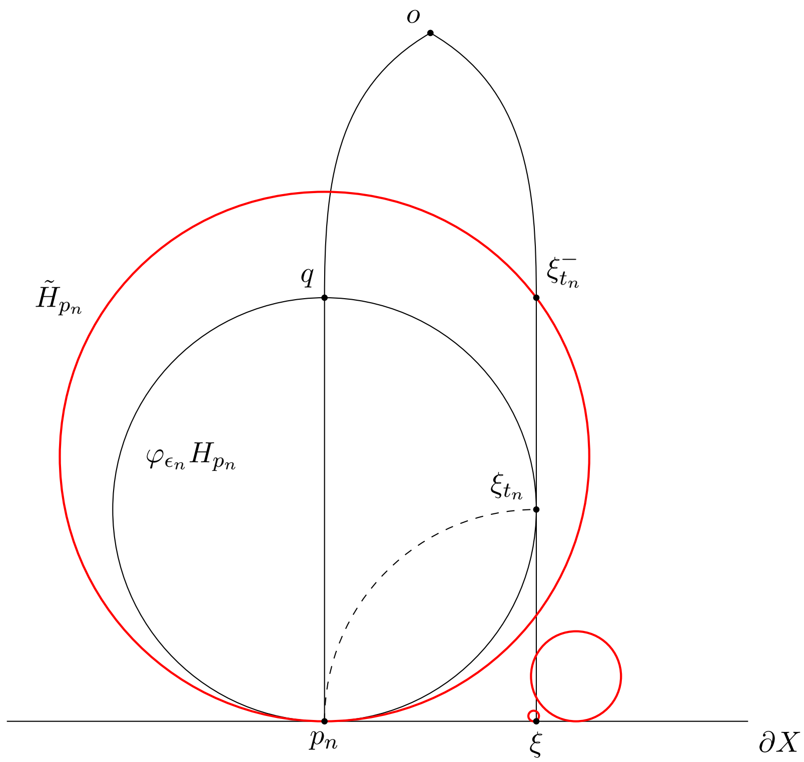

FIGURE 6. For the proof of Theorem 6.10. The red horoballs correspond to the collection of horoballs $\tilde{H}_p$ which have been rescaled by $O(\alpha)$ so that if some choice of geodesic $[o, \xi)$ cuts a horoball H_p, then any other choice of geodesic cuts $\tilde{H}_p$.

See that $\log r_{p_n}^{-1} \leq t_n + O(\alpha)$ because by Corollary 2.4 $\log r_{p_n}^{-1} + O(\alpha)$ is the distance from o to the horoball $\tilde{H}_{p_n}$, which contains the point ξ_{t_n}. Also, note that by Corollary 2.4 and the definition of the horoball $\varphi_{\epsilon_n} H_{p_n}$, the distance from o to the horoball $\varphi_{\epsilon_n} H_{p_n}$ is $-\log(r_{p_n} \varphi_{\epsilon_n}(r_{p_n})) + O(\alpha)$.

Let $\xi_{t_n^-}$ denote the entry point of the geodesic $[o, \xi)$ in the horoball $\tilde{H}_{p_n}$. First, since each r_p is chosen so that the union of all H_p is the non-cuspidal part, $\xi^-_{t_n}$ is within uniform bounded distance of Γo, so there exists C_1 such that

$$d(\xi_{t_n}, \partial H_{p_n}) - C_1 \leq d(\xi_{t_n}, \Gamma o) \leq d(\xi_{t_n}, \xi_{t_n^-}) + C_1. \tag{6.10}$$

By Corollary 2.9, since ξ_{t_n} is a closest point projection of p_n onto $[o, \xi)$,

$$d(\xi_{t_n}, \xi_{t_n^-}) = \beta_{p_n}(\xi_{t_n^-}, \xi_{t_n}) + O(\alpha). \tag{6.11}$$

Moreover, for any x on the boundary of H_{p_n} we have, since Busemann functions are 1-Lipschitz, $d(x, \xi_{t_n}) \geq \beta_{p_n}(x, \xi_{t_n}) = \beta_{p_n}(\xi_{t_n^-}, \xi_{t_n}) + O(\alpha)$, hence

$$d(\xi_{t_n}, \partial H_{p_n}) \geq \beta_{p_n}(\xi_{t_n^-}, \xi_{t_n}) + O(\alpha). \tag{6.12}$$

Finally, since ξ_{t_n} is on the boundary of $\varphi_{\epsilon_n} H_{p_n}$ and $\xi_{t_n^-}$ is on the boundary of $\tilde{H}_{p_n}$, by the quasi-cocycle property of Busemann functions and the definition of horospheres we have

$$\beta_{p_n}(\xi_{t_n^-}, \xi_{t_n}) = -\log \varphi_{\epsilon_n}(r_{p_n}) + O(\alpha). \tag{6.13}$$

Thus, combining Equations (6.10), (6.11), and (6.13) yields the estimate

$$\begin{aligned}
d(\xi_{t_n}, \Gamma o) &\le \beta_{p_n}(\xi_{t_n^-}, \xi_{t_n}) + C_1 + O(\alpha) \\
&= -\log \varphi_{\epsilon_n}(r_{p_n}) + C_1 + O(\alpha) \\
&= \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})}\right) \log\left(\log(r_{p_n}^{-1})\right) + C_1 + O(\alpha) \\
&\le \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})} \log(t_n + O(\alpha))\right) + C_1 + O(\alpha). \quad (6.14)
\end{aligned}$$

On the other hand, let q be the closest point to o on $[o, p_n) \cap H_{p_n}(r_{p_n})$. Since q lies on the boundary of the horoball, by the quasi-cocycle property of the Busemann function,

$$-\log \varphi_{\epsilon_n}(r_{p_n}) = \beta_{p_n}(q, \xi_{t_n}) + O(\alpha).$$

Since ξ_{t_n} is a closest point projection of p_n onto $[o, \xi)$, Corollary 2.9 and the quasi-cocycle property of Busemann functions gives us that

$$t_n + \log(\varphi_{\epsilon_n}(r_{p_n})) = d(o, \xi_{t_n}) - \beta_{p_n}(q, \xi_{t_n}) + O(\alpha) = d(o, q) + O(\alpha).$$

Thus, by Corollary 2.4 we obtain

$$t_n + \log(\varphi_{\epsilon_n}(r_{p_n})) \le \log r_{p_n}^{-1} + C_2 \quad (6.15)$$

where C_2 is a constant depending only on the hyperbolicity constant. Thus, using Equations (6.10), (6.12), (6.13) and (6.15),

$$\begin{aligned}
d(\xi_{t_n}, \Gamma o) &\ge -\log(\varphi_{\epsilon_n}(r_{p_n})) - C_1 - O(\alpha) \\
&= \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})}\right) \log(\log r_{p_n}^{-1}) - C_1 - O(\alpha) \\
&\ge \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})}\right) \log(t_n + \log(\varphi_{\epsilon_n}(r_{p_n})) - C_2) - C_1 - O(\alpha) \\
&= \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})}\right) \log\left(t_n - \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})}\right) \log(\log(r_{p_n}^{-1})) - C_2\right) - C_1 - O(\alpha) \\
&\ge \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})}\right) \log\left(t_n - \left(\frac{1+\epsilon_n}{2(\delta-\delta_{\max})}\right) \log(t_n - O(\alpha)) - C_2\right) - C_1 - O(\alpha).
\end{aligned}$$

Thus, noting that $\epsilon_n \to 0$ as $n \to \infty$,

$$\frac{1}{2(\delta-\delta_{\max})} \le \limsup_{t \to +\infty} \frac{d(\xi_t, \Gamma o)}{\log t}.$$

It remains to prove the upper bound on the limsup. For values of t such that $\xi_t \in X_{nc}$, the result is trivial. Recall that each t_n is chosen so that for all values t so that $\xi_t \in H_{p_n}(r_{p_n})$, the distance $d(\xi_t, \Gamma o)$ is maximized up to $O(\alpha)$ at $t = t_n$. Then for such $t \ge t_n$, $\frac{d(\xi_t, \Gamma o)}{\log(t)} \le \frac{d(\xi_{t_n}, \Gamma o)}{\log(t_n)}$ as desired by Equation (6.14). Now consider $t \le t_n$. Then, applying Equations (6.10), (6.11), (6.12),

$$\begin{aligned}
d(\xi_{t_n}, \Gamma o) &\ge \beta_{p_n}(\xi_{t_n^-}, \xi_{t_n}) - C_1 - O(\alpha) = d(\xi_{t_n}, \xi_{t_n^-}) - C_1 - O(\alpha) \\
&= |t_n^- - t_n| - C_1 - O(\alpha) \ge t_n - t - C_1 - O(\alpha).
\end{aligned}$$

Thus, $t \ge t_n - d(\xi_{t_n}, \Gamma o) - C_1 - O(\alpha)$, and by Equation (6.14),

$$\frac{d(\xi_t, \Gamma o)}{\log(t)} \le \frac{d(\xi_{t_n}, \Gamma o)}{\log(t_n - d(\xi_{t_n}, \Gamma o) - C_1 - O(\alpha))} \le \frac{d(\xi_{t_n}, \Gamma o)}{\log(t_n - C_3 \log(t_n) - C_3)}$$

for some constant $C_3 > 0$. The result follows Equation (6.14). □

7. Applications to Hilbert geometry

In this section, we will apply the results to a class of geometries called *Hilbert geometries.* These geometries generalize hyperbolic geometry to a non-Riemannian setting in which the metric is not CAT(0) [Egl97, Appendix B] but, for a large family of examples of interest, is Gromov hyperbolic. We first introduce the preliminary background.

A domain Ω in real projective space $\mathbb{R}P^n$ is *properly convex* if there exists an affine chart in which Ω is bounded and convex, meaning its intersection with any line segment is connected. We say Ω is *strictly convex* if, moreover, the projective boundary $\partial_{\text{proj}}\Omega$ in an affine chart does not contain any open line segments. Any properly convex domain admits a natural, projectively invariant metric called the *Hilbert metric* which is central to this application. The Hilbert metric is defined as follows. Choose an affine chart in which Ω is bounded; then for each $x, y \in \Omega$, any projective line passing through x and y must intersect $\partial_{\text{proj}}\Omega$ at exactly two points, a, b. Then

$$d_\Omega(x,y) := \frac{1}{2}\big|\log[a;x;y;b]\big|$$

where $[a;x;y;b] := \frac{|a-y||b-x|}{|a-x||b-y|}$ is the cross-ratio with respect to the ambient affine metric inherited from the chart. The normalization factor of $\frac{1}{2}$ ensures that if Ω is an ellipsoid, then (Ω, d_Ω) is the Beltrami–Klein model for hyperbolic space of constant curvature -1.

The cross-ratio is a projective invariant, hence the metric does not depend on the chart, and projective transformations which preserve Ω are isometries with respect to d_Ω. Straight lines are geodesics for this metric, and are the only geodesics when Ω is strictly convex. Evidently, the Hilbert metric is proper and the topological boundary $\partial_{\text{proj}}\Omega$ in $\mathbb{R}P^n$ is a compactification of Ω on which projective transformations that preserve Ω act as homeomorphisms. If $\Gamma < \text{PSL}(n+1,\mathbb{R})$ preserves Ω and is discrete then its action for the Hilbert metric is properly discontinuous. Thus, the definition of geometrical finiteness and all the related notions (Section 3, Definition 3.1) are coherent for the action of a discrete group of projective transformations Γ on Ω. The limit set Λ_Γ is again the smallest closed invariant set, and is hence basepoint independent, when $|\Lambda_\Gamma| \geq 3$ and Ω is strictly convex with C^1 boundary (for more, see [CM14a, Définition 4.1, Lemme 4.2]). We note the following lemma:

Lemma 7.1. *Let $\Omega \subset \mathbb{R}P^n$ be strictly convex and $\Gamma < \text{PSL}(n+1,\mathbb{R})$ a discrete group preserving Ω. If the convex hull C_Γ is a hyperbolic metric space when endowed with the Hilbert metric, then $\Lambda_\Gamma \subset \partial_{proj}\Omega$ is naturally identified with the hyperbolic boundary of C_Γ.*

Proof. Fix a basepoint $o \in C_\Gamma$ to define Λ_Γ. Recall the hyperbolic boundary ∂C_Γ is the set of geodesic rays at o up to bounded equivalence. See that by strict convexity of Ω and the definition of the Hilbert metric, if two projective line segments starting at o and going to $\partial_{\text{proj}}\Omega$ are bounded distance apart then they coincide. Thus, the map $\partial C_\Gamma \to \Lambda_\Gamma$ defined by associating to each geodesic ray based at o and contained in C_Γ its unique intersection with $\partial_{\text{proj}}\Omega$ is well-defined. This point of intersection must lie in Λ_Γ by definition of C_Γ. By convexity of C_Γ, the map is surjective. □

A pair (Ω, Γ) where $\Gamma < \text{PSL}(n+1,\mathbb{R})$ is a discrete group that preserves Ω is called a *convex real projective structure* on the quotient manifold Ω/Γ, and when Ω is strictly convex, we specify that the structure is a *strictly convex real projective structure.*

7.1. Relation to work of Crampon–Marquis.

Geometrical finiteness in Hilbert geometry was first studied by Crampon–Marquis [CM14a]. They showed, for example, that when Ω is strictly convex with C^1 boundary, the isometries of (Ω, d_Ω) can be classified as elliptic, parabolic, loxodromic and quasi-loxodromic as in Section 3 [CM14a, Theorem 3.3, Section 3.5]. Crampon–Marquis used two definitions of geometrical finiteness:

Definition 7.2 (Crampon–Marquis [CM14a])**.** For a strictly convex $\Omega \subset \mathbb{R}P^n$ with C^1 boundary and a non-elementary discrete group $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$ which preserves Ω, the action of Γ is *geometrically finite on* $\partial_{\mathrm{proj}}\Omega$ if every point ξ in Λ_Γ is either a bounded parabolic point or a conical limit point. This is the same as Definition 3.1 and so we will say that in this case, Γ is a *geometrically finite* group. More strongly, Crampon-Marquis define Γ to be *geometrically finite with hyperbolic cusps* if every point ξ in Λ_Γ is either a bounded parabolic point with stabilizer conjugate into $\mathrm{SO}(n, 1)$ or a conical limit point. Crampon–Marquis refer to geometrical finiteness with hyperbolic cusps as "Γ acting geometrically finitely on Ω". We will avoid this language to reduce confusion with Definition 3.1.

Crampon–Marquis show that these two conditions are not equivalent. In [CM14a, Proposition 10.7], they produce a group Γ in $\mathrm{PSL}(5, \mathbb{R})$ which preserves a strictly convex set Ω with C^1 boundary in $\mathbb{R}P^4$ such that Γ is geometrically finite, but it is not geometrically finite with hyperbolic cusps.

7.2. **Examples with hyperbolic convex hull.** There is a large family of examples which are geometrically finite with hyperbolic cusps, and these examples will have hyperbolic convex hull:

Theorem 7.3 ([CM14a, Théorème 1.8])**.** *If Ω is strictly convex with C^1 boundary and Γ is geometrically finite with hyperbolic cusps, then Γ is relatively hyperbolic, and the convex hull C_Γ with the Hilbert metric is a hyperbolic metric space.*

On a strictly convex Ω with C^1 boundary, any group acting with cofinite volume, and more generally any geometrically finite group for which all parabolic stabilizers have maximal rank, will be geometrically finite with hyperbolic cusps ([CLT15, Theorem 0.4, Theorem 0.5], [CM14a, Théorème 7.14]). In fact when Ω admits a finite volume quotient, it is enough to assume that either Ω is strictly convex or Ω has C^1-boundary since these criteria are equivalent in that case [CLT15, Theorem 0.15].

More explicitly, examples include all geometrically finite $\Gamma < \mathrm{PSL}(3, \mathbb{R})$ preserving a strictly convex $\Omega \subset \mathbb{R}P^2$ with C^1 boundary. There are many such actions: for instance, the moduli space of finite volume strictly convex real projective structures on a surface of genus g with p punctures has real dimension $16g - 16 + 8p$ [Mar10], and contains the $6g - 6 + 2p$ dimensional Teichmüller space via the Beltrami–Klein model. In higher dimensions, the moduli space of finite volume strictly convex real projective structures can be nontrivial even though the Teichmüller space is trivial. In every dimension, there are deformable examples [BM16, Mar12] via the Johnson-Millson bending construction [JM87]. In dimension three, there are deformable examples that arise from a generalization of Thurston's gluing equations [BC21]. There are also examples of closed topological manifolds that admit a strictly convex projective structure but do not admit a Riemannian constant curvature hyperbolic metric [Ben06, Kap07]. It is plausible that there is a corresponding finite volume non-compact example which admits a strictly convex real projective structure but does not admit a metric of constant negative curvature. Our results apply to any such examples.

The convex hull can be hyperbolic even when the action of Γ only satisfies the weaker, standard notion of geometrical finiteness as in Definition 3.1 without having hyperbolic cusps. For instance, in the above-mentioned example ([CM14a, Proposition 10.7]), the convex hull is hyperbolic [DGK21, Zim21]. It seems plausible that for any Hitchin representation of a geometrically finite Fuchsian group which preserves a properly convex subset of $\mathbb{R}P^d$, there exists some, possibly different, strictly convex set Ω with C^1 boundary preserved by Γ such that the convex hull of the limit set in $\partial_{\mathrm{proj}}\Omega$ is a hyperbolic metric space, but at the moment it is not known (see [CZZ21] for more details).

Remark 7.4. Let us note that [CM14a, Théorème 9.1] also claims that hyperbolicity of the convex hull C_Γ of Γ implies Γ is geometrically finite with hyperbolic cusps, but as discussed above, that is not true. However, we do not need this implication in this paper, so this is irrelevant for our purposes. Corrections to [CM14a] are expected in a forthcoming erratum by Blayac–Marquis.

On the other hand, one might optimistically hope that whenever Γ is geometrically finite, the convex hull is hyperbolic. However, in the same forthcoming article, Blayac–Marquis produce examples such that Γ is geometrically finite and fails to have hyperbolic convex hull in Ω. Interestingly, for the same provided examples, they produce another Γ-invariant Ω' for which the convex hull is hyperbolic. Whether or not this phenomenon holds in general is unclear.

7.3. **Patterson–Sullivan measures for geometrically finite Hilbert geometries.** Crampon showed in his thesis that Patterson's construction can be adapted to the setting of geometrically finite groups with hyperbolic cusps when Ω is strictly convex with C^1 boundary [Cra11, Theorem 4.2.1]. We call a measure arising from this construction a *Patterson–Sullivan measure*. Crampon proves the measures are supported on the limit set [Cra11, Section 4.2.1], and then proves in the case of surfaces that the Patterson–Sullivan measures have no atoms [Cra11, Lemma 4.3.3, Proposition 4.3.5]. These arguments generalize to higher dimensions due to [CM14b, Corollaire 7.18], which generalizes [Cra11, Lemma 1.3.4]. In recent work, Zhu confirms that these results extend to higher dimensions in the strictly convex with C^1 boundary setting (see [Zhu20, Lemma 11, Proposition 12, Corollary 13]). These results hinge on a calculation that any bounded parabolic group preserving Ω with rank r and conjugate into $\mathrm{SO}(n,1)$ has critical exponent $\delta_\Pi = \frac{r}{2}$, and if Π is a subgroup of a geometrically finite group Γ, then $\delta_\Pi < \delta_\Gamma$ [Zhu20, Lemma 11], [CM14b, Lemme 9.8]. The work of Zhu was further generalized by Blayac to the rank one setting, without the strictly convex with C^1 boundary condition [Bla21, Theorem 1.6] and by Blayac–Zhu when Γ is geometrically finite and Ω is strictly convex with C^1 boundary [BZ23, Theorem 9.1, Lemma 9.13, Proposition 9.14]. Blayac–Zhu elaborate after [BZ23, Theorem 5.4] on why finiteness of Patterson–Sullivan measure given by [BZ23, Theorem 9.1] implies that Patterson–Sullivan measure has full support on Λ_Γ.

7.4. **Growth independence of domain.** We observe in this section that the critical exponent and the upper and lower annular growth rates do not depend on the domain. One consequence of this is if Γ is geometrically finite with hyperbolic cusps, then all parabolic subgroups have exponential growth and their critical exponent is equal to half of the rank of the group, as for hyperbolic space. We will not need this observation as our applications will be more general.

For $G \in \mathrm{SL}(d,\mathbb{R})$, let $\mu_1(G), \dots, \mu_d(G)$ be the singular values of G, listed in decreasing order. Then for $g \in \mathrm{PSL}(d,\mathbb{R})$, define $\kappa(g) := \frac{1}{2}(\log \mu_1(G) - \log \mu_n(G))$ for any lift G of g.

Proposition 7.5 (Proposition 10.1 [DGK21])**.** *For any properly convex domain Ω in $\mathbb{R}P^n$ and any $o \in \Omega$, there exists a constant C such that for all $g \in \mathrm{Aut}(\Omega)$,*

$$|d_\Omega(o, go) - \kappa(g)| \leq C.$$

Lemma 7.6. *When $\Pi < \mathrm{PSL}(n+1,\mathbb{R})$ preserves some properly convex domain Ω, the upper and lower annular growth rates $\delta_\Pi^-, \delta_\Pi^+$ and the critical exponent δ_Π do not depend on Ω. In particular, if Ω' is another properly convex domain in $\mathbb{R}P^n$ and $\Pi < \mathrm{Aut}(\Omega) \cap \mathrm{Aut}(\Omega')$, then the action of Π on Ω has δ-tempered growth if and only if the action of Π on Ω' has δ-tempered growth.*

Proof. By Proposition 7.5, fixing an $o \in \Omega$ and $o' \in \Omega'$, we have $d_\Omega(o, go) = d_{\Omega'}(o', go') + O(1)$ where $g \in \Pi$. Let

$$B_\Omega(t) = \#\{g \in \Pi \mid d_\Omega(o, go) \leq t\}, \qquad B_{\Omega'}(t) = \#\{g \in \Pi \mid d_\Omega(o', go') \leq t\},$$

and

$$A_{\Omega,r}(t) = \frac{1}{r} \log \frac{B_\Omega(t+r)}{B_\Omega(t)}.$$

Then there exists a constant C such that

$$B_{\Omega'}(t - C) \leq B_\Omega(t) \leq B_{\Omega'}(t + C).$$

It follows that δ_Π does not depend on Ω. Similarly, for $r > 2C$,

$$\frac{r-2C}{r} A_{\Omega', r-2C}(t+C) \leq A_{\Omega,r}(t) \leq \frac{r+2C}{r} A_{\Omega', r+2C}(t-C).$$

It is then straightforward to verify that $\delta_\Pi^\pm$ is independent of Ω. □

7.5. **Growth of parabolic subgroups.** We prove in this section that parabolic subgroups in strictly convex Hilbert geometry have mixed exponential growth, as defined in Definition 3.6. As a consequence, we will see that if Γ is geometrically finite and preserves a strictly convex domain with C^1 boundary, then Γ has tempered parabolic subgroups.

Proposition 7.7. *Let Ω be a strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary. Then every discrete parabolic subgroup of* $\mathrm{PSL}(n+1,\mathbb{R})$ *preserving Ω has mixed exponential growth for the Hilbert metric.*

Proof. Let Π be a parabolic subgroup in $\mathrm{Aut}(\Omega)$. Then by [BZ23, Proposition 9.6] (which is a consolidation of [CM14a, Proposition 7.1] and [CM14a, Lemme 7.6]), Π is a uniform lattice in its Zariski closure $\mathcal{N}$ and moreover, $\mathcal{N}$ can be written as $\mathcal{N} = K \times U$ where K is compact and U is unipotent. If one considers the projection $p_U : \mathcal{N} = K \times U \to U$, the image $\Pi' = p_U(\Pi)$ is a lattice in U, and the kernel of the restriction of p_U to Π is finite.

Fix on G the norm $\|g\| := \mathrm{tr}(g^t g)^{1/2}$, which is submultiplicative. Let us introduce, for $g \in G$, the notation

$$|g| := \frac{\log \|g\| + \log \|g^{-1}\|}{2}.$$

Since the norm is submultiplicative, we have $|g| \geq 0$ for any g. Moreover, we also have

$$|gh| \leq |g| + |h| \qquad \text{for any } g, h \in G.$$

Notice by Proposition 7.5, using that all matrix norms are equivalent, that

$$d_\Omega(o, go) = |g| + O(1) \qquad \text{for any } g \in \Pi. \tag{7.1}$$

Let $\mathfrak{u}$ denote the Lie algebra of U. The exponential map $\exp : \mathfrak{u} \to U$ is a diffeomorphism, and the pushforward of a Lebesgue measure on $\mathfrak{u}$ is the Haar measure on U. Let $P : \mathfrak{u} \to \mathbb{R}$ be given by

$$P(x) = \|\exp(x)\|^2 \|\exp(-x)\|^2.$$

Note that $\log P(x) = 4|\exp(x)|$. Since the norm is submultiplicative, $P(x) \geq C > 0$ for all $x \in \mathfrak{u}$ where $C = \|\mathrm{Id}\|^2$ is a constant depending only on n. Since U is a unipotent matrix group, $\mathfrak{u}$ is a set of nilpotent matrices with bounded degree. Then by the definition of matrix exponentiation, the entries of $\exp(x)$ are polynomials in the entries of x, and it then follows from the definition that P is a polynomial in $\dim(\mathfrak{u})$ many variables. Note that P is proper since exp is a diffeomorphism and the norm function is a proper map for any choice of matrix norm on the finite dimensional vector space $\mathfrak{u}$. Letting λ denote Lebesgue measure on $\mathfrak{u}$, see that the pushforward of λ by exp is Haar measure on U (see e.g. [CG90, Theorem 1.2.10]).

Now, by Benoist-Oh [BO07, Corollary 7.3(a)], there exist $a \in \mathbb{Q}_{\geq 0}, b \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda(\{x \in \mathbb{R}^d \ : \ P(x) \leq t\}) \asymp t^a (\log t)^b \qquad \text{for any } t > 0.$$

We will use this to show that

$$\#\{g \in \Pi' \ : \ |g| \leq t\} \asymp e^{4at} t^b \qquad \text{for any } t > 0. \tag{7.2}$$

Let F be a compact fundamental domain for the action of Π' on U, which exists since Π' is a uniform lattice. Let m be the Haar measure on U. If $\Delta := \sup\{|f|, f \in F\}$ is the diameter of F, then

$$m(\{u \in U \ : \ |u| \leq t - \Delta\}) \leq m(F)\#\{g \in \Pi' \ : \ |g| \leq t\} \leq m(\{u \in U \ : \ |u| \leq t + \Delta\}).$$

Moreover, since the pushforward of the Lebesgue measure under the exponential map is the Haar measure,

$$m(\{u \in U \ : \ |u| \le t\}) = \lambda(\{x \in \mathfrak{u} \ : \ P(x) \le e^{4t}\}).$$

Equation (7.2) follows.

Finally, we show that, for any $t > 0$,

$$\#\{g \in \Pi \ : \ d_\Omega(o, go) \le t\} \asymp e^{4at} t^b. \tag{7.3}$$

Let K_0 be the kernel of $p_U|_\Pi$, c the cardinality of K_0, and $A = \sup\{d_\Omega(o, ko) : k \in K_0\}$. Then, since any $g \in \Pi$ can be written as $g = ku$ for $k \in K$ and $u \in \Pi'$, and at most c values of g correspond to a given value of u,

$$\#\{g \in \Pi \ : \ |g| \le t - A\} \le \#\{u \in \Pi' \ : \ |u| \le t\} \le c\#\{g \in \Pi \ : \ |g| \le t + A\}$$

hence, for any $t > 0$,

$$\#\{g \in \Pi \ : \ |g| \le t\} \asymp e^{4at} t^b.$$

Finally, Equation (7.1) now implies Equation (7.3), as desired. □

We will now see how to compute the growth rate for rank one parabolic subgroups.

Lemma 7.8. *Let $\Omega \subset \mathbb{R}P^n$ be a properly convex domain and $\Pi < \mathrm{PSL}(n+1, \mathbb{R})$ a discrete group preserving Ω. If Π is a parabolic group of rank one, generated (up to finite index) by g, then Π has pure exponential growth, i.e. for any $T > 0$ one has*

$$\#\{h \in \Pi \ : \ d_\Omega(o, ho) \in [T, T+1]\} \asymp e^{\delta_\Pi T}$$

with $\delta_\Pi = \frac{1}{k-1}$, where k is the size of the largest Jordan block of g.

For example, Lemma 7.8 applies to the Crampon–Marquis example [CM14a, Proposition 10.7] with $n = k = 4$.

Proof of Lemma 7.8. By [CLT15, Proposition 2.13], which applies to any properly convex Ω, every eigenvalue of g is equal to 1. Thus by taking the Jordan form, g is conjugate to a unipotent matrix. By taking powers, we have

$$\|g^n\| \asymp |n|^{k-1} \qquad \text{for any } n \in \mathbb{Z}$$

where k is the size of the largest Jordan block of g and $\|\cdot\|$ is any invariant norm, such as the leading singular value. Then by Proposition 7.5 (see also [BZ23, Proposition 2.6]), we obtain

$$d_\Omega(o, g^n o) = \frac{\log \|g^n\| + \log \|g^{-n}\|}{2} + O(1) = (k-1)\log n + O(1)$$

thus

$$\#\{n \in \mathbb{Z} \ : \ d_\Omega(o, g^n o) \in [T, T+1]\} \asymp e^{\frac{T}{k-1}}$$

for any $T > 0$, which proves the claim. □

From the previous two results, we obtain that parabolic subgroups have tempered growth.

Corollary 7.9. *Let $\Omega \subset \mathbb{R}P^n$ be a strictly convex domain with C^1 boundary, and $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$ a geometrically finite group preserving Ω. Then Γ has tempered parabolic subgroups.*

Proof. By Proposition 7.7 we obtain $\delta_\Pi^\pm = \delta_\Pi$; since every parabolic group contains a rank one parabolic subgroup, from Lemma 7.8 we obtain that $\delta_\Pi > 0$. Blayac–Zhu prove [BZ23, Lemma 8.13] that for any non-elementary group Γ acting on a strictly convex Ω with C^1 boundary, any parabolic subgroup $\Pi < \Gamma$ has $\delta_\Pi < \delta_\Gamma$. This concludes the proof. □

7.6. **Statement of result.** We are now ready to verify that geometrically finite Hilbert geometries satisfy the global shadow lemma and logarithm law. The theorem below applies to all the examples discussed in Subsection 7.2, and implies Theorems 1.2 and 1.6 from the introduction.

Theorem 7.10. *Let Ω be a strictly convex domain in $\mathbb{R}P^n$ with C^1 boundary and $\Gamma < \mathrm{PSL}(n+1, \mathbb{R})$ a geometrically finite group which preserves Ω. Assume the convex hull of the limit set C_Γ is hyperbolic with respect to the Hilbert metric. Then any Patterson–Sullivan measure μ satisfies the global shadow lemma (Theorem 5.1), and for μ-a.e. $\xi \in \Lambda_\Gamma$,*

$$\limsup_{t \to +\infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \frac{1}{2(\delta - \delta_{\max})}$$

where $\delta_{\max}$ is the maximal growth rate of any parabolic subgroup, and ξ_t is the point on the geodesic ray $[o, \xi)$ that is distance t from o.

Proof. We need only verify the hypotheses. First, (C_Γ, d_Ω) is a proper hyperbolic metric since the Hilbert metric is proper on Ω. Then C_Γ has boundary Λ_Γ by Lemma 7.1, and Γ acts minimally on Λ_Γ since the action is non-elementary, as discussed in the beginning of this section. The Patterson–Sullivan measures constructed by Blayac–Zhu are a conformal density of dimension δ_Γ on Λ_Γ with no atoms [BZ23, Proposition 9.14, Theorem 9.1] (see Subsection 7.3 for elaboration), and Γ has δ_Γ-tempered parabolic subgroups by Corollary 7.9. Thus, the hypotheses of the global shadow lemma (Theorem 5.1) are satisfied. Finally, since Γ has mixed exponential growth by Proposition 7.7, the hypotheses of the logarithm law (Theorem 6.10) are satisfied, completing the proof. □

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