A 0-1 law and cusp excursion for geometrically finite actions on coarsely hyperbolic metric spaces

Joint with Giulio Tiozzo

Khinchin's Theorem 1926 : $\psi : \mathbb{N} \to \mathbb{R}^+$ monotone decr (maybe other hypotheses).

Let $\Theta(\psi) := \{ x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{\psi(q)}{q}$ for ∞-ly many $\frac{p}{q} \in \mathbb{Q} \}$

Then

$$\sum_{q \in \mathbb{N}} \psi(q) = \infty \Rightarrow \Theta(\psi) \text{ has full measure}$$

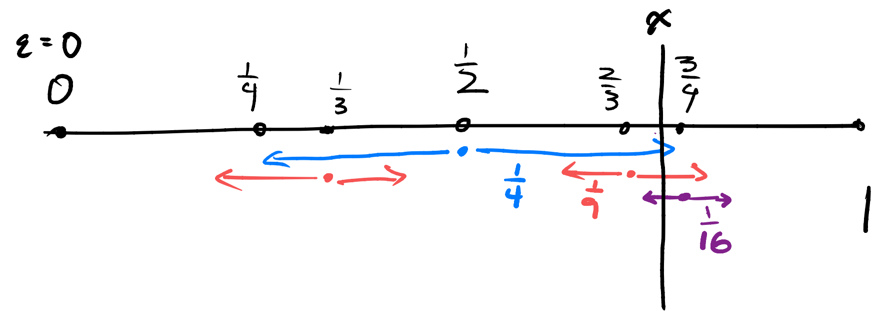
and

$$\sum_{q \in \mathbb{N}} \psi(q) < \infty \Rightarrow \Theta(\psi) \text{ has measure zero}$$

restrict to $[0,1]$ to get a "0-1" law

Application $\psi_\varepsilon(q) = \frac{1}{q^{1+\varepsilon}}$

$\Theta(\psi) = \{ x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$ for ∞-ly many $\frac{p}{q} \in \mathbb{Q} \}$



so for this x is in 3 of these balls. Will it be in ∞-ly many such balls?
If $\varepsilon > 0$ then the balls get even smaller

by Khinchin:

$$\sum_{q \in \mathbb{N}} \psi_\varepsilon(q) = \sum_{q \in \mathbb{N}} \frac{1}{q^{1+\varepsilon}} \begin{cases} = \infty & \text{for } \varepsilon = 0 \\ < \infty & \text{for } \varepsilon > 0 \end{cases}$$

hence

$$\Theta(\psi_\varepsilon) \begin{cases} \text{has full measure for } \varepsilon = 0 \\ \text{has measure zero for } \varepsilon > 0 . \end{cases}$$

In our picture, with probability 1, x is in ∞-ly many balls of radius $\frac{1}{q^2}$.

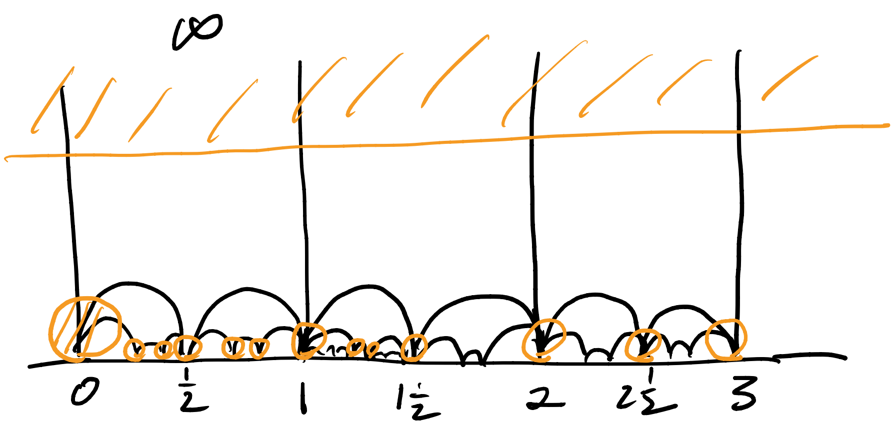
horoball packings for the hyperbolic plane

by example

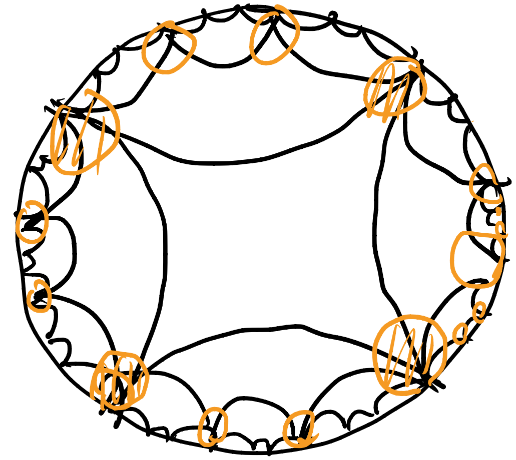
$\Gamma \curvearrowright \mathbb{H}^2$ finite area

tiling of $\mathbb{H}^2$

$\rightsquigarrow$ horoball packing (Γ-equivariant)



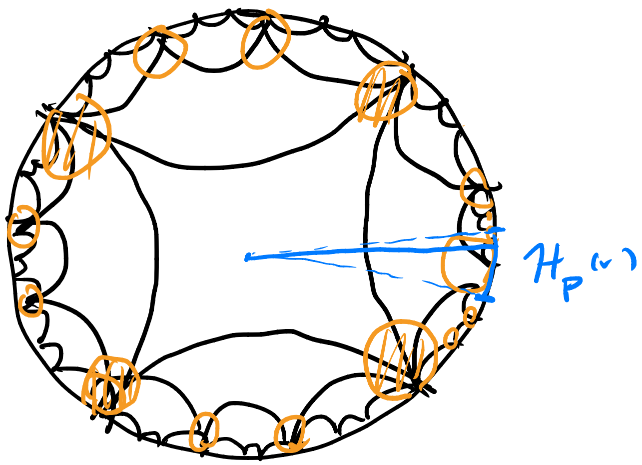
Define horoball packing?



$\mathcal{P}$ = { centers of horoballs in the packing }

r_p = Euclidean radius of the horoball centered at p from a fixed original packing

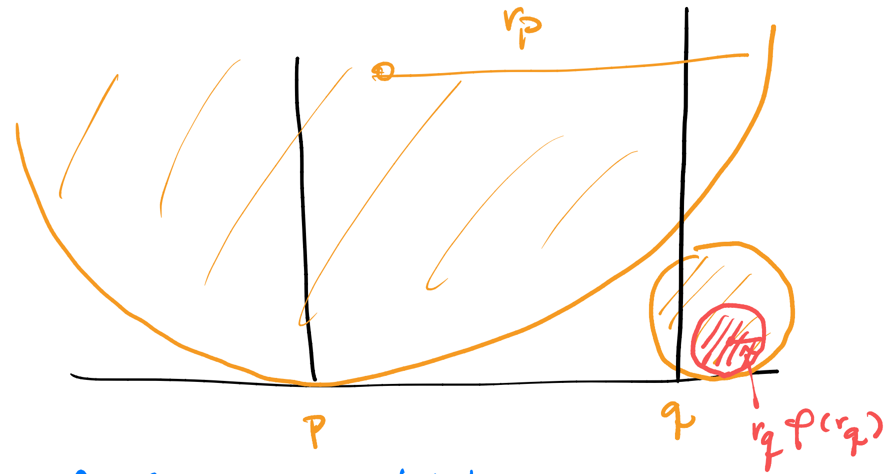
$H_p(r)$ = shadow of the horoball centered at p with radius r.



radius of shadow $H_p(r)$ in S^1 is $\sim r$

- $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ increasing (so that $\varphi(r)$ decr. as $r \to 0$)

- $\Theta(\varphi) = \{ x \in \partial \mathbb{H}^2 : x \in H_p(r_p \varphi(r_p))$ for ∞-ly many $p \in \mathcal{P} \}$



$\varphi \equiv 1 \Rightarrow$ no shrinking

in this case, a.e. x in ∞-ly many shadows.

$\varphi < 1 \Rightarrow$ shrinking, x may no longer be in ∞-ly many shadows

Thm: (Stratmann-Velani, Sullivan)

[Khinchin-type Theorem] for small $\lambda < 1$,

$\sum_{n \in \mathbb{N}} \varphi(\lambda^n) < \infty \iff \Theta(\varphi)$ has Leb measure zero

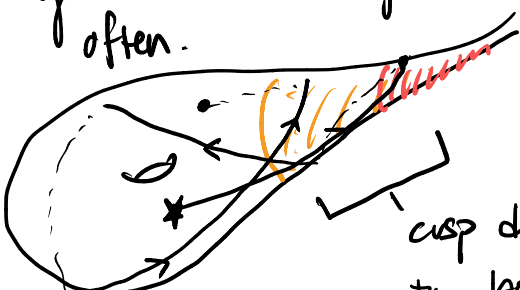
$= \infty \iff \Theta(\varphi)$ has Leb measure one

Application to cusp excursion

Horoball packing projects to neighborhood of the cusp.

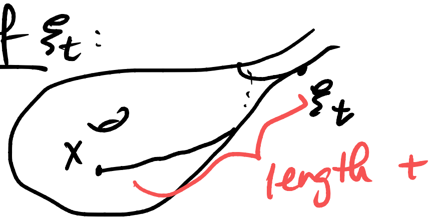
φ shrinks neighborhoods.

if they stay the "same size" ($\varphi \equiv 1$), a.e. geodesic visits neighborhood ∞-ly often.



cusp depth = distance to boundary of original horoball

defn of ξ_t:



Thm (Stratmann-Velani, Sullivan)

$\boxed{\text{[Logarithm Law] For a.e. } \xi \in S^1, \quad \limsup_{t \to \infty} \frac{\text{cusp depth}(\xi_t)}{\log(t)} = 1.}$

with Tiozzo, we generalize these to the setting of coarsely hyp. metric spaces and geom. fin. grp actions

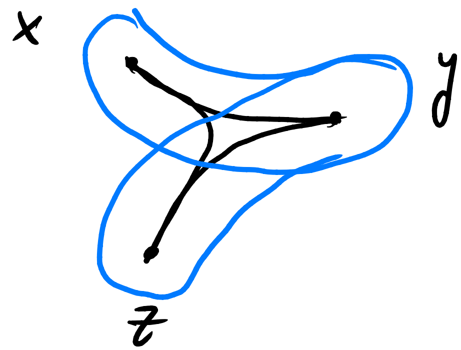
Defn

For functions $f, g : \mathcal{U} \to \mathbb{R}$

- $f \approx g$ iff $\exists\, c$ s.t. on $\mathcal{U}$,
$$f - c \le g \le f + c$$
- $f \overset{\sim}{\asymp} g$ iff $\exists\, c$ s.t. on $\mathcal{U}$,
$$\tfrac{1}{c} g \le f \le c g$$

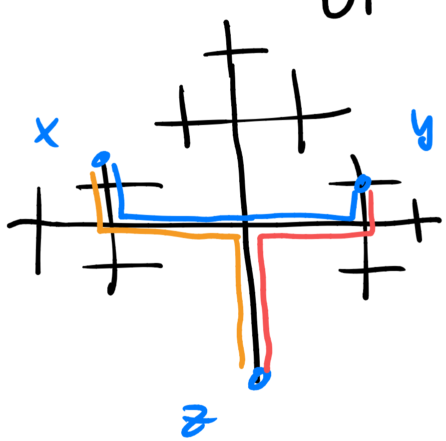
Recall: hyp metric spaces

(X,d) δ-hyperbolic if



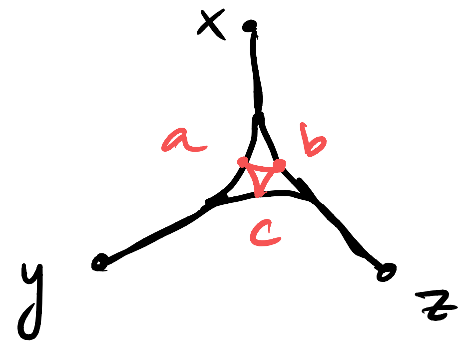
(X,d) Gromov hyp or coarsely hyp if δ-hyp some $\delta>0$

e.g. trees are 0-hyperbolic



when (X,d) coarsely hyp, "triangles are tripods"

Alt defn: diam(inner triangle) $\leq \delta$



Note: for a tree, inner Δ = pt

Fact: "geodesics are Morse" meaning

$$d(x,[y,z]) \approx \frac{1}{2}\Big(d(x,z)+d(x,y) - d(y,z)\Big)$$

=: Gromov product $\langle y,z\rangle_x$

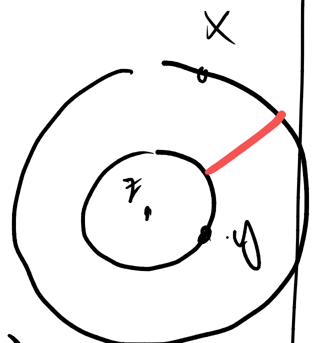
(X,d) hyp metric space

fix $o \in X$.

$\partial X := \{ \text{geodesic rays based at } o \} \big/ \text{bndd equiv}$

How to get horoball packing?

Defn Busemann functions



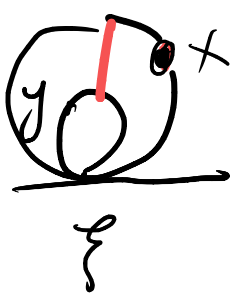
$x, y, z \in X$,

$$\beta_z(x,y) := d(x,z) - d(y,z)$$

Notice $|\beta_z(x,y)| \le d(x,y)$.

for $\xi \in \partial X$, define

$$\beta_\xi(x,y) := \liminf_{z \to \xi} \beta_z(x,y)$$



Defn:

Fix $o \in X$. horoball of radius r centered at ξ

$$H_\xi(r) := \{ x \in X : \beta_\xi(x,o) \le \log r \}$$

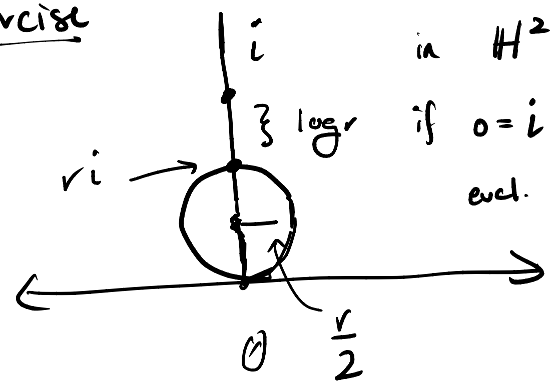
Horosphere $= \log r$

example $\mathbb{R}^2$

example $\mathbb{H}^2$ β_ξ is a limit

recover usual horoballs

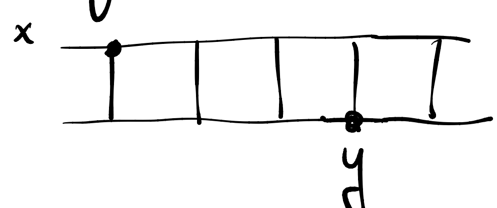
exercise



in $\mathbb{H}^2$: if $o = i$, $\xi = 0$ then eucl. radius of $H_\xi(r) = \frac{r}{2}$

In a hyp metric space $\beta_\xi(x,y)$ is not a limit

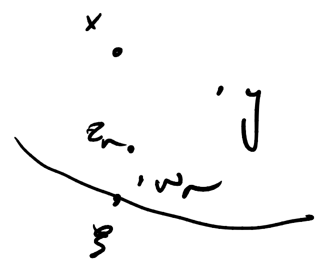
e.g. (exercise?) ladder is coarsely hyp



$\xi = +\infty$

In a hyp metric space $\beta_\xi(x,y)$ is coarsely well-defined

<u>idea</u>: $z_{t_n}, w_{t_n} \to \xi$.

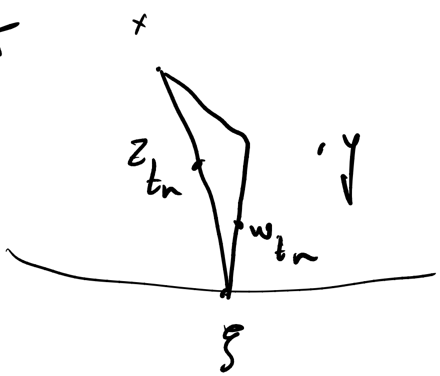


<u>exercise</u>

choose to lie on geodesics from x to ξ dist. t_n away

want $\beta_{z_n}(x,y) \approx \beta_{w_n}(x,y)$.

see that



$$|\beta_{z_{t_n}}(x,y) - \beta_{w_{t_n}}(x,y)|$$

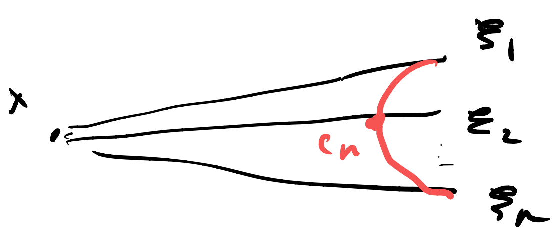
$$= |d(y,z_{t_n}) - d(y,w_{t_n})|$$

$$\leq d(z_{t_n}, w_{t_n}) \leq C$$

by defn of ∂X

C is uniform by hyp of X

<u>Pf</u>:



$d(\xi_1(t), \xi_n(t)) \not< n \ \forall t$.

$\Rightarrow c_n \to c \in X$.

Properties of Busemann functions:

- 1-Lipshitz

$$|\beta_\xi(x,y)| \leq d(x,y)$$

- Γ-inv

$$\beta_{\gamma\xi}(\gamma x, \gamma y) = \beta_\xi(x,y)$$

- cocycle

$$\beta_\xi(x,y) + \beta_\xi(y,z) \approx \beta_\xi(x,z)$$

Def $\mathcal{H}_\xi(r)$ = shadow of $H_\xi(r)$

$= \{ \eta \in \partial X$ such that some geod. o to η intersects $H_\xi(r) \}$

Defn (Recall) Define properly discts

Γ properly discts, non-elem geometrically finite if every pt in Λ_Γ is either conical or bdd parabolic.

Γ convex cocompact if only conical.

$\boxed{\text{From now on:}}$

Assume $\Gamma < \mathrm{Isom}\, X$ is prop. discts, non-elem and acting geom. fin. on X

$\Lambda_\Gamma :=$ limit set $= \overline{\Gamma o} \setminus \Gamma o \subseteq \partial X$.

C_Γ = convex hull of Λ_Γ.

Fact (Tukia, Yaman, Bowditch)

- fin. many conjugacy classes $\Pi_1, \dots, \Pi_d$ parabolic subgroups
- Γ hyp relative to $\{\Pi_1, \dots, \Pi_d\}$

Prop: (Bowditch)

$\Gamma \curvearrowright X$ geom. fin. $\Rightarrow$ $\exists$ quasi-Γ-inv., pairwise disjoint, horoball packing of X $\{H_p(r_p)\}_{p \in \mathcal{P}}$, and

$$\Gamma \curvearrowright C_\Gamma - \cup H_p(r_p) \quad \text{cocompact}$$

Defn: $\Pi < \Gamma$ parabolic subgp has mixed exponential growth if $\exists\, a_\Pi, \delta_\Pi \geq 0$ s.t. for $t \geq 0$,

$$B_\Pi(t) := \#\{ g \in \Pi : d(o, go) \leq t \} \asymp e^{\delta_\Pi t} (t+1)^{a_\Pi}$$

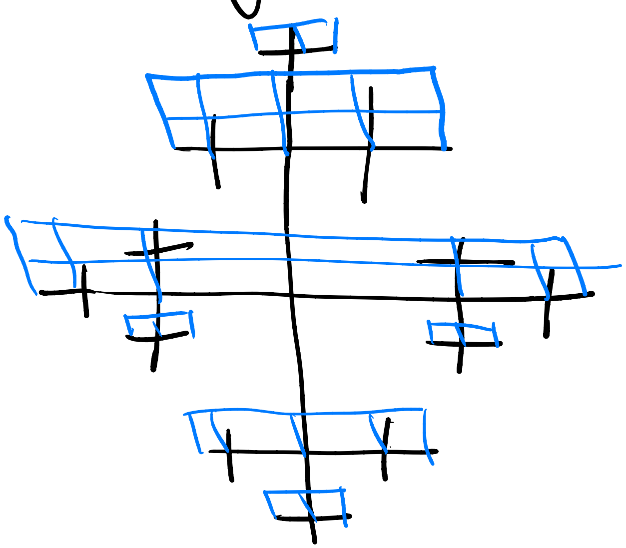
understanding mixed exp. growth

Exercise: $\pi \curvearrowright \mathbb{H}^n$

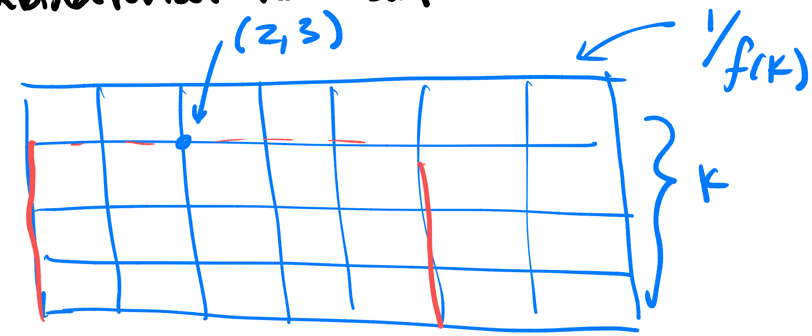
$a_\pi = 0 \qquad \delta_\pi = \frac{\operatorname{rank} \pi}{2}$

Recall Madeline's discussion of the Groves-Manning cusp space:

$Cay(F_2)$



combinatorial horoball



e.g. $f(k) = 2^{-k}$ $\qquad 2^{-5} = \frac{1}{32}$

$d((0,0),(0,32)) \leq 2 \times 5 + 32 \times 2^{-5}$

$= 11 < 32$

$X = Cay(F_2) \cup$ combinatorial horoballs

d = induced metric by f

Thm (G-M, Hruska-Healy, ...) for certain f

w/ $e^{at} \leq f(t) \leq e^{bt}$ for some $a, b > 0$

then (X, d) a hyperbolic metric space

and $\Gamma = F_2$ acts geometrically finitely

exercise in that case, $B_\pi(t) \asymp f(t)$

Can create mixed exp. growth or anything.

Khinchin-type Theorem

Let $\Pi_1, \dots, \Pi_d$ be the finitely many parabolic subgroups of Γ up to conjugation.

$$P^i = \Gamma . p_i \quad \text{where} \quad \Pi_i p_i = p_i$$

Defn: $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ Khinchin if φ incr and $\exists$ $b_1 < 1$, $b_2 > 0$ such that

$$\varphi(b_1 x) \geq b_2 \varphi(x)$$

$\forall x \in \mathbb{R}^+$ (important only for small x)

Defn:
$$\Theta^i(\varphi) = \{ x \in \Lambda_\Gamma : x \in \mathcal{H}_p(r_p \varphi(r_p)) \text{ for } \infty\text{-ly many } p \in P^i \}$$

Khinchin series

$$K^i_\lambda(\varphi) = \sum \varphi(\lambda^n)^{2(\delta_\Gamma - \delta_{\Pi_i})} (-2 \log \varphi(\lambda^n) + 1)^{a_{\Pi_i}}$$

for $i = 1, \dots, d$

where
$$\delta_\Gamma = \limsup_{t \to \infty} \frac{1}{t} \log \#\{ \gamma \in \Gamma : d(o, \gamma o) \leq t \}.$$

Thm (B.-Tiozzo)

[Khinchin-type theorem] more next time

μ Patterson-Sullivan measure*

Γ mixed exp growth

for any Khinchine function φ,

(1) $\mu(\Theta_\lambda(\varphi)) = 0$ if $K_\lambda(\varphi) < \infty$

(2) $\mu(\Theta_\lambda(\varphi)) = 1$ if $K_\lambda(\varphi) = \infty$.

For $S = \mathbb{H}^2/\Gamma$ finite area,

$$\delta_\Gamma = 1, \quad \delta_{\Pi_i} = \frac{1}{2}, \quad a_{\Pi_i} = 0$$

hence $K^i_\lambda(\varphi) = \sum \varphi(\lambda^n)$.

Kleinian (Bishop-Jones, Sullivan)

$$\delta_\Gamma = \text{Haus.dim } \Lambda_\Gamma$$

Thm (B.-Tiozzo)

[Logarithm law] Same μ.

$$\delta_\Pi^* = \max \delta_{\Pi_i}$$

For μ-a.e. $\xi \in \Lambda_\Gamma$,

$$\limsup_{t \to \infty} \frac{\text{cusp depth}(\xi_t)}{\log t} = \frac{1}{2(\delta_\Gamma - \delta_\Pi^*)}.$$

end day 1

__Note:__ recovers $\mathbb{H}^2$ case, finite area

__Note:__ the $2(\delta_\Gamma - \delta_\Pi^*)$ is coming from the fine scaling properties of μ.

__Prior results__

- Sullivan / Stratmann-Velani geom. fin $\Gamma \curvearrowright \mathbb{H}^n$
- Parlin-Hersonsky X Riem. pinched neg. curvature and $a_{\Pi_i} = 0$ $\forall i = 1, \dots, d$

__Future results__

(Horton) filtration by each conjugacy class of parabolic subgroups

__Application__

by Benoist-Oh as in Blayac-Zhu,

Π discrete parabolic preserves strictly cvx Ω with C^1 boundary

$\Rightarrow$ mixed exponential growth.

Thm applies to (Ω, d_Ω) Hilbert metric if Gromov hyperbolic

__Growth condition__ is actually more general for the Khinchin-type theorem, but not for log law.

Day 2

Fix:

$\Gamma \curvearrowright X$ non-elem, geom. fin

(X,d) proper geodesic coarsely hyp

Let $\Pi_1, \dots, \Pi_d$ be the finitely many parabolic subgroups of Γ up to conjugation.

$\mathcal{P} = \{ p \in \Lambda_\Gamma \text{ parabolic f.p.s} \}$

For $\Gamma' < \Gamma$, define

$$B_{\Gamma'}(t) = \{ g \in \Gamma' : d(o, go) \le t \}.$$

$\Pi < \Gamma$ has <u>mixed exp growth</u> if

$$B_\Pi(t) \asymp e^{\delta_\Pi t} (t+1)^{a_\Pi}$$

some $0 < \delta_\Pi$, $0 \le a_\Pi$.

$H_p(r) = \{ x \in X : \beta_p(o,x) \le \log r \}$

<u>Bowditch</u> $\exists$ quasi-Γ-inv horoball packing $\{ H_p(r_p) \}_{p \in \mathcal{P}}$

$\mathrm{stab}_\Gamma(p) \curvearrowright H_p \approx H_p$

and

$\Gamma \curvearrowright C(\Lambda_\Gamma) \setminus \bigcup H_p(r_p)$

cocompactly.

$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ decr, "Khinchin"

$\Theta(\varphi) = \{ x \in \Lambda_\Gamma : x \in H_p(r_p \varphi(r_p))$ for ∞-ly many $p \in \mathcal{P} \}$

<u>Khinchin series</u>

$$K_\lambda(\varphi) = \sum_{i=1}^{d} \sum_{n \in \mathbb{N}} \varphi(\lambda^n)^{2(\delta_\Gamma - \delta_{\Pi_i})} (-2\log \varphi(\lambda^n) + 1)^{a_{\Pi_i}}$$

some $0 < \lambda < 1$

Thm (B.-Tiozzo)

[Khinchin-type Theorem] more next time

μ Patterson-Sullivan measure*

Γ mixed exp growth, $\delta_\Pi < \delta_\Gamma$

for any Khinchine function φ,

(1) $\mu(\Theta_\lambda(\varphi)) = 0$ if $K_\lambda(\varphi) < \infty$

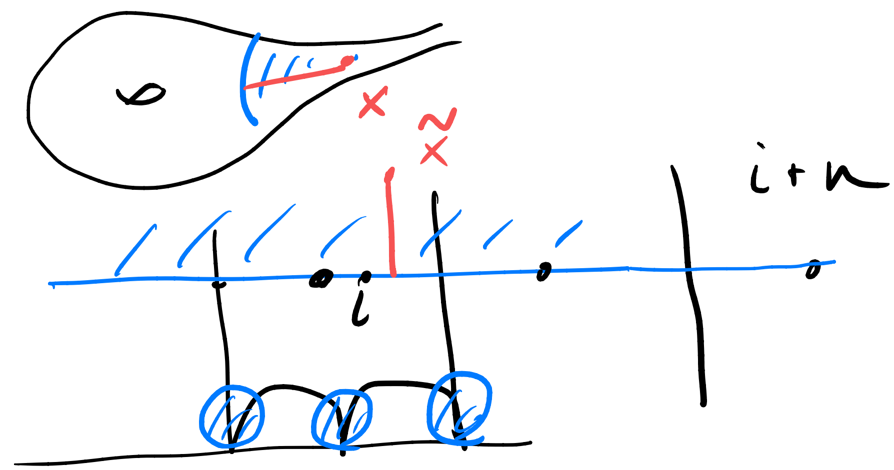
(2) $\mu(\Theta_\lambda(\varphi)) = 1$ if $K_\lambda(\varphi) = \infty$.

Thm (B.-Tiozzo)

[Logarithm law] Same μ. $\delta_\Pi^* = \max \delta_{\Pi_i}$

For μ-a.e. $\xi \in \Lambda_\Gamma$,

$$\limsup_{t\to\infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \frac{1}{2(\delta_\Gamma - \delta_\Pi^*)}.$$



Γo does not enter horoballs

$d(\Gamma o, C_\Gamma - \bigcup_{p\in P} H_p(r_p))$ bndd

$\Rightarrow$ cusp depth(x) $\sim$

$$= d(x, C_\Gamma - \bigcup_{p\in P} H_p(r_p))$$

$$\approx d(x, \Gamma o). \qquad \square$$

Thm (Coornaert)

$$B_\Gamma(t) \asymp e^{\delta_\Gamma t}$$

for some δ_Γ which we call the critical exponent

goal: the $2(\delta_\Gamma - \delta_\pi^*)$ is coming from the fine scaling properties of μ.

Prior results

- Sullivan / Stratmann-Velani geom. fin $\Gamma \curvearrowright \mathbb{H}^n$
- Parlin-Hersonsky X Riem. pinched neg. curvature and $a_{\pi_i} = 0$ $\forall i = 1, \ldots, d$

Future results

(Horton) filtration by each conjugacy class of parabolic subgroups

Growth condition is actually more general for the Khinchin-type theorem,

our proof of log law needs mixed exp growth but Madeline is generalizing

Application

certain relatively Anosov rep$^{\text{n}}$s

recall $S = \mathbb{H}^2/\Gamma$ means

$\Gamma \hookrightarrow PSL(2,\mathbb{R})$

any hyp Γ!

Can study $\Gamma \hookrightarrow PSL(d,\mathbb{R})$

In some cases, natural Γ-inv set in $\mathbb{RP}^d$ is Gromov-hyp

Benoist-Oh, Blayac-Zhu: has mixed exp growth

many examples $a_\pi \neq 0$ but cannot realize any explicitly.

Kim-Oh: others act on GM-cusp space.

Patterson-Sullivan construction

from Didac

Recall the critical exponent of Γ is

$$\delta_\Gamma = \limsup_{t \to \infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : d(o, \gamma o) \le t\}.$$

e.g. $\#\{d(o,\gamma o) \le t\} = e^{2t} \Rightarrow \delta_\Gamma = 2$

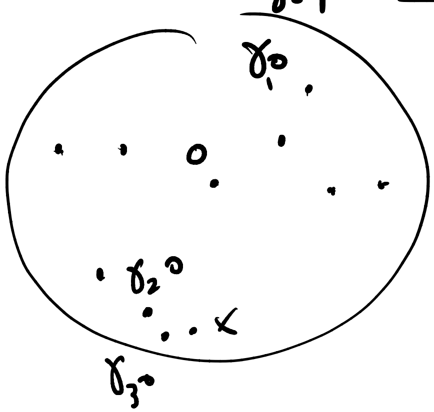
exercise δ_Γ = absissa of convergence for

$$P(o,x,s) := \sum_{\gamma \in \Gamma} e^{-s\, d(x, \gamma o)}$$

(first for $x = o$, then triangle ineq.)

Defn measure on $\overline{X}$

$$\mu_{x,s} = \sum_{\gamma \in \Gamma} \frac{e^{-s\, d(x,\gamma o)}}{P(o,o,s)} \delta_{\gamma o}$$



up to extraction, define Patterson-Sullivan measure

$$\mu_x := \lim_{s \searrow \delta_\Gamma}{}^{*} \mu_{x,s}$$

when Γ is divergent type

$$P(o,x,\delta_\Gamma) = \infty$$

$\operatorname{supp} \mu_x = \Lambda_\Gamma$.

when Γ is not divergent type, modify w/ h which does not change δ_Γ.

see that $\forall$ compact $K \subseteq X$,

$$\mu_{x,s}(K) \xrightarrow[s \searrow \delta_\Gamma]{} 0$$

and $\forall$ open $O \subseteq \overline{X} \setminus \overline{C}_\Gamma$,

$$\mu_{x,s}(O) = 0.$$

Let $\mu := \mu_o$.

defn $\{\nu_x\}_{x\in X}$ is a δ-quasi-conformal density at ∞: $\|\nu_x\|<\infty$ and

- $\gamma_* \nu_x = \nu_{\gamma x}$ Γ-inv
- $\frac{d\nu_x}{d\nu_y}(\xi) \asymp e^{-\delta \beta_\xi(x,y)}$ transf. rule

Thm: $\{\mu_x\}_{x\in X}$ is δ_Γ-q.c. density

Patterson: Fuchsian
Sullivan: Kleinian

Coornaert: non-elementary, proper, coarsely hyperbolic

Let $\mu := \mu_0$

Pf ideas

for $s > \delta_\Gamma$

$$g_* \mu_{x,s}(A) = \mu_{x,s}(g^{-1}A)$$

$$= \sum_{\gamma\in\Gamma} \frac{e^{-s\,d(x,\gamma o)}}{P(o,o,s)} \delta_{\gamma o}(g^{-1}A)$$

$\gamma o \in g^{-1}A \iff g\gamma o \in A$

$$= \frac{1}{P(o,o,s)} \sum_{\gamma\in\Gamma} e^{-s\,d(gx, g\gamma o)} \delta_{g\gamma o}(A)$$

$$= \frac{1}{P(o,o,s)} \sum_{g\gamma\in\Gamma} e^{-s\,d(gx, g\gamma o)} \delta_{g\gamma o}(A)$$

$$\to \mu_{gx}$$

transf. rule

$\xi = \lim \gamma_n o$

$\frac{d\mu_x}{d\mu_y}(\xi)$

$$\frac{d\mu_{x,s}}{d\mu_{y,s}}(\gamma_n o) = \frac{P(0,0,s)}{P(0,0,s)} \frac{e^{-s\, d(x, \gamma_n o)}}{e^{-s\, d(y, \gamma_n o)}}$$

$$= e^{-s(d(x,\gamma_n o) - d(y, \gamma_n o))}$$

$\xrightarrow{\text{coarsely}}$! $e^{-\delta_\Gamma \beta_\xi(x,y)}$

side bar:

Patterson-Sullivan current:

$$d\mu_{PS}(\xi,\eta) = e^{-\delta_p \langle \xi, \eta \rangle_o} d\mu(\xi)\, d\mu(\eta)$$

my fave current

exercise? $\Gamma \curvearrowright \mathbb{H}^2$ cocompact

$$d\mu_{PS}(\xi,\eta) = \frac{1}{(\xi-\eta)^2} d\lambda(\xi,\eta)$$

on $\mathbb{R}^2 \setminus \{\xi = \eta\}$

Fact = Liouville

Notn: Shadow of a ball radius r

$\mathcal{O}_r(x,y) = \{\xi \in \partial X$ s.t. $\exists$ geodesic ray $[x,\xi)$ intersecting $B(y,r)$. $\}$

shadows generate topology on ∂X for fixed r.

Sullivan's Shadow Lemma Γ non-elem, $x \in X$, $\{\mu_x\}$ δ_Γ-q.c., $r \gg 0$,

$$\frac{1}{c} e^{-\delta_\Gamma d(x,\gamma x)} \le \mu_x(\mathcal{O}_r(x,\gamma x)) \le c e^{-\delta_\Gamma d(x,\gamma x)}$$

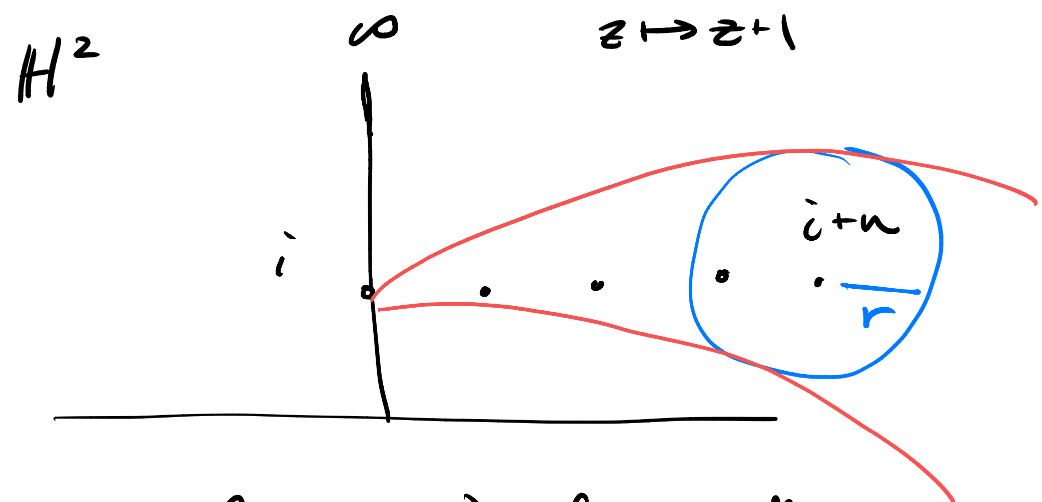
Cor: $\xi \in \Lambda_\Gamma$ conical, then $\mu(\{\xi\}) = 0$

Pf: $\gamma_n x \to \xi$ conically $\Rightarrow$ $d(x,\gamma x) \to \infty$ and for n large, $\xi \in \mathcal{O}_r(x,\gamma_n x)$ hence

$$\mu(\{\xi\}) \le Ce^{-d(x,\gamma_n x)} \to 0$$

Note does not work for parabolic fixed points

$\mathbb{H}^2$ $\quad$ $z \mapsto z+1$

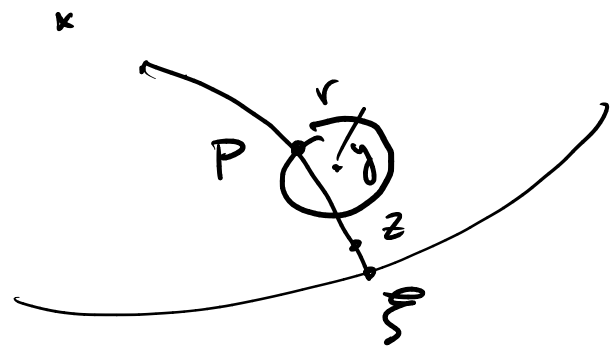


$\infty \notin \mathcal{O}_r(i, i+n)$ for n suff. large pf of Shadow lemma (Roblin)

Pf: Lemma: $\forall\, \xi \in \mathcal{O}_r(x,y)$,

$$d(x,y) - 2r \le \beta_\xi(x,y) \le d(x,y)$$

Pf:



$$d(x,y) \le d(x,p) + r$$
$$d(y,z) \le d(z,p) + r$$

so

$$\begin{aligned} & d(x,z) - d(y,z) \\ &= d(x,p) + d(p,z) - d(y,z) \\ &\geq d(x,y) - r + d(y,z) - r - d(y,z). \end{aligned}$$

□

By quasi-Γ-invariance,

$$\mu_x(\mathcal{O}_r(x,\gamma x)) = \mu_x(\gamma\, \mathcal{O}_r(\gamma^{-1}x, x)) \asymp \mu_{\gamma^{-1}x}(\mathcal{O}_r(\gamma^{-1}x, x)) \quad (*)$$

and the transf. rule,

$$\frac{1}{c}\int_{\mathcal{O}_r(\gamma^{-1}x,x)} e^{-\delta_\Gamma \beta_\xi(\gamma^{-1}x,x)}\, d\mu_x(\xi) \le * \le c\int_{\mathcal{O}_r(\gamma^{-1}x,x)} e^{-\delta_\Gamma \beta_\xi(\gamma^{-1}x,x)}\, d\mu_x(\xi)$$

Then 1-Lipschitz and Lemma $\Rightarrow$

$$\frac{1}{c}\int_{\mathcal{O}_r(\gamma^{-1}x,x)} e^{-\delta_\Gamma d(\gamma^{-1}x,x)}\, d\mu_x(\xi) \le * \le c\int_{\mathcal{O}_r(\gamma^{-1}x,x)} e^{-\delta_\Gamma (d(\gamma^{-1}x,x) - 2r)}\, d\mu_x(\xi).$$

Then

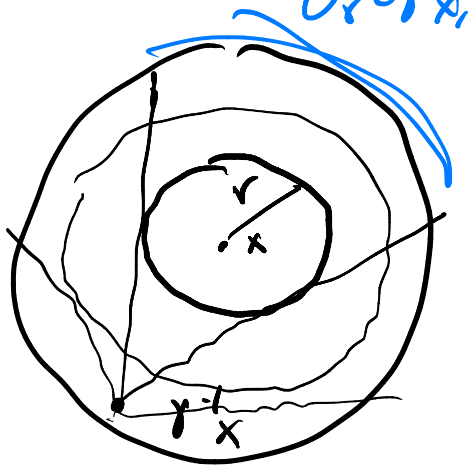
$$* \leq e^{2\delta_\Gamma r} e^{-\delta_\Gamma d(x,\gamma x)} \|\mu_x\|$$

completes the upper bound.

For the lower bound, we have

$$* \geq e^{-\delta_\Gamma d(x,\gamma x)} \mu_x(\mathcal{O}_r(\gamma^{-1}x, x))$$

$\mathcal{O}_r(\gamma^{-1}x,x)$



How small could $\mu_x(\mathcal{O}_r(\gamma^{-1}x,x))$ be?

if r large, use ping-pong argument to show $\forall x, \exists O_1, O_2$ open in Λ_Γ s.t. $\forall y \in X$, one of $O_i \subseteq \mathcal{O}_r(y,x)$. $\square$

Cor: δ_Γ-q.c. densities equiv. for convex cocompact

Pf: $\{\mu_x\}, \{\nu_x\}$

$$\frac{1}{C^2} \leq \frac{\mu_x(\mathcal{O}_r(x,\gamma x))}{\nu_x(\mathcal{O}_r(x,\gamma x))} \leq C^2$$

$\forall \xi \in \Lambda_\Gamma$, $\exists$ seq. $\gamma_n x \to \xi$ conically again, so $\forall n$ large

$\xi \in \mathcal{O}_r(x, \gamma_n x)$ hence

$$\frac{d\mu_x}{d\nu_x}(\xi) \in \left[\tfrac{1}{C^2}, C^2\right] \subseteq (0,\infty) \quad \forall \xi.$$

Talk #3

Last time (Rem: typo)

$\mathcal{O}_r(x,y) = \{\xi \in \partial X$ s.t. $\exists$ geodesic ray $[x,\xi)$ intersecting $B(y,r).\}$

$\{\mu_x\}_{x\in X}$ Patterson-Sullivan

Sullivan's Shadow Lemma Γ non-elem, $x\in X$, $\{\mu_x\}$ δ_Γ-q.c, $r \gg 0$,

$$\frac{1}{C} e^{-\delta_\Gamma d(x,\gamma x)} \le \mu_x(\mathcal{O}_r(x,\gamma x)) \le C e^{-\delta_\Gamma d(x,\gamma x)}$$

when Γ convex cocompact, $\delta_\Gamma = \text{Hausdim}\, \Lambda_\Gamma$ (Patterson)

Rem: Ricks, Gekhtman-Ma

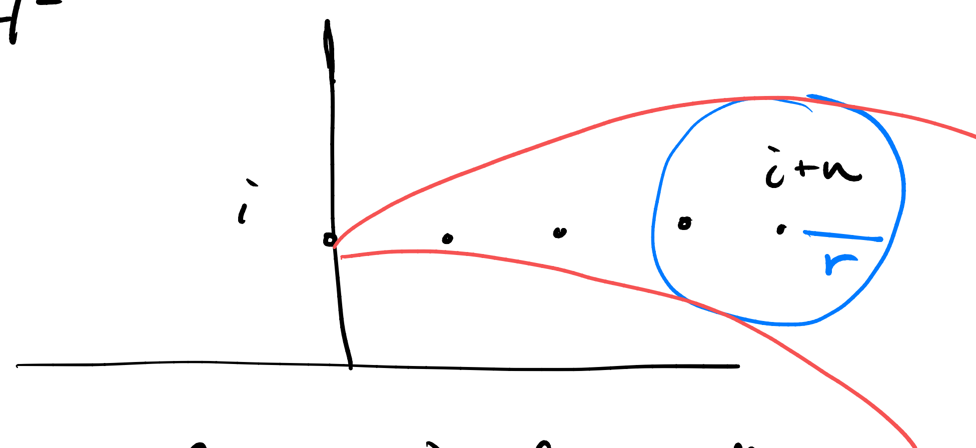
Cor: $\xi \in \Lambda_\Gamma$ conical, then $\mu(\{\xi\}) = 0$

Pf: $\gamma_n x \to \xi$ conically $\Rightarrow$ $d(x,\gamma_n x) \to \infty$ and for n large, $\xi \in \mathcal{O}_r(x,\gamma_n x)$ hence

$$\mu(\{\xi\}) \le C e^{-d(x,\gamma_n x)} \to 0$$

Note does not work for parabolic fixed points

$\mathbb{H}^2$ $\qquad \infty \qquad z \mapsto z+1$



$\infty \notin \mathcal{O}_r(i, i+n)$ for n suff. large

Cor: δ_Γ - q.c. densities equiv. for convex cocompact Γ

Pf: $\{\mu_x\}, \{\nu_x\}$

$$\frac{1}{C^2} \le \frac{\mu_x(\mathcal{O}_r(x,\gamma x))}{\nu_x(\mathcal{O}_r(x,\gamma x))} \le C^2$$

$\forall\, \xi \in \Lambda_\Gamma$, $\exists$ seq. $\gamma_n x \to \xi$ conically again, so $\forall\, n$ large $\xi \in \mathcal{O}_r(x, \gamma_n x)$ hence

$$\frac{d\mu_x}{d\nu_x}(\xi) \in \left[\tfrac{1}{C^2}, C^2\right] \subseteq (0,\infty) \quad \forall\, \xi.$$

Defn $A \subseteq \Lambda_\Gamma$ is Γ-inv if $\forall\, \gamma \in \Gamma$, $\gamma A = A$.

Defn: μ is ergodic for $\Gamma \curvearrowright \partial X$ if every Γ-invariant set has trivial μ-measure, meaning it is null or co-null.

e.g. δ_γ where γ closed orbit for geodesic flow

Let

Cor: μ is ergodic for Γ cvx compt $\curvearrowright \Lambda_\Gamma$

Pf of cor: Assume $A \subseteq \Lambda_\Gamma$ is Γ-inv and $\mu(A) > 0$. Then define

$$\mu_A(E) := \mu(E \cap A)$$

see that μ_A is also a q.c. density:

- quasi-Γ-invariant

$$\gamma_* \bar{\mu}_x(B) = \bar{\mu}_x(\gamma^{-1}B)$$

$$= \mu_x(A \cap \gamma^{-1}B)$$

$$= \mu_x(\gamma^{-1}A \cap \gamma^{-1}B)$$

$$= \mu_x(\gamma^{-1}(A \cap B))$$

$$= \mu_{\gamma x}(A \cap B)$$

$$= \bar{\mu}_{\gamma x}(B)$$

- transformation rule

open $B_n \downarrow \xi$

$$\frac{d\bar{\mu}_y}{d\bar{\mu}_x}(\xi) = \lim_{n\to\infty} \frac{\mu_y(B_n \cap A)}{\mu_x(B_n \cap A)}$$

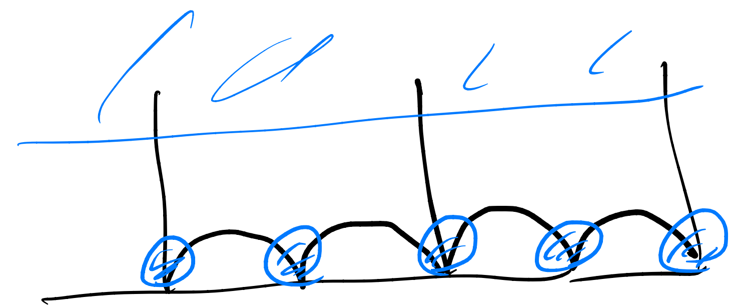
$$= \lim_{n\to\infty} \frac{\int_{B_n\cap A} e^{-\delta_\Gamma \beta_\xi(x,y)}\, d\mu_x(\xi)}{\mu_x(B_n \cap A)}$$

$$\to e^{-\delta_\Gamma \beta_\xi(x,y)}.$$

Then $\{\mu_x\}, \{\bar{\mu}_x\}$ equivalent

hence $\mu_x(A^c) = \bar{\mu}_x(A^c) = 0$

for Γ geom. fin,



Global Shadow Lemma / fluctuating density theorem

Sullivan, Stratmann-Velani for hyp
Paulin-Hersonsky, Schapira for neg. curvature

B.-Tiozzo (X,d) hyp, Γ geom. fin. $\boxed{\text{one } \Pi < \Gamma}$ $\delta_\Pi < \delta_\Gamma$. Then

$$\mu(\mathcal{O}_r(o,y)) \asymp e^{-\delta_\Gamma d(o,y)} e^{(2\delta_\Pi - \delta_\Gamma) d(y, \Gamma o)}$$

Note: $y \notin \cup H_p(r_p) \Rightarrow \asymp e^{-\delta_\Gamma d(o,y)}$

Cor: μ has no atoms

Pf: $d(o,y) \geq d(y, \Gamma o)$

$$\Rightarrow \mu(\mathcal{O}_r(o,y)) \leq e^{2\delta_\Pi - 2\delta_\Gamma d(y, \Gamma o)}$$

$\to 0$ as $y \to \xi \in \partial X$

since $\delta_\Pi < \delta_\Gamma$.

Note: $y \notin \cup H_p(r_p)$

recover usual shadow lem.

Matsuzaki - Yabuki - Jaerisch:

Patterson - Sullivan measure exists and is ergodic

pictures, how is this different from the original.

Next: say something now about upgrading from standard shadow lemma

Idea one piece of the argument: where does orbit growth come in

Credit for strategy to Schapira

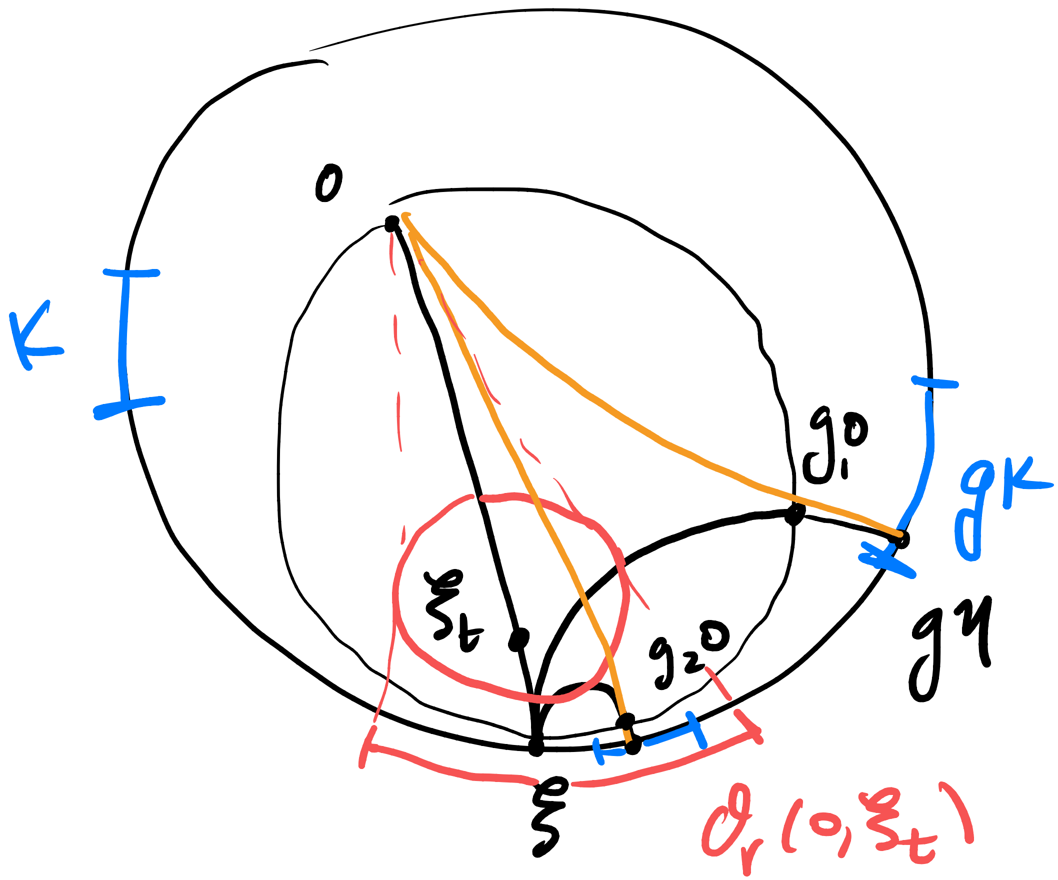
$\Pi = \mathrm{Stab}_\Gamma \xi$, ξ_t = pt dist. t on geod $[o, \xi)$

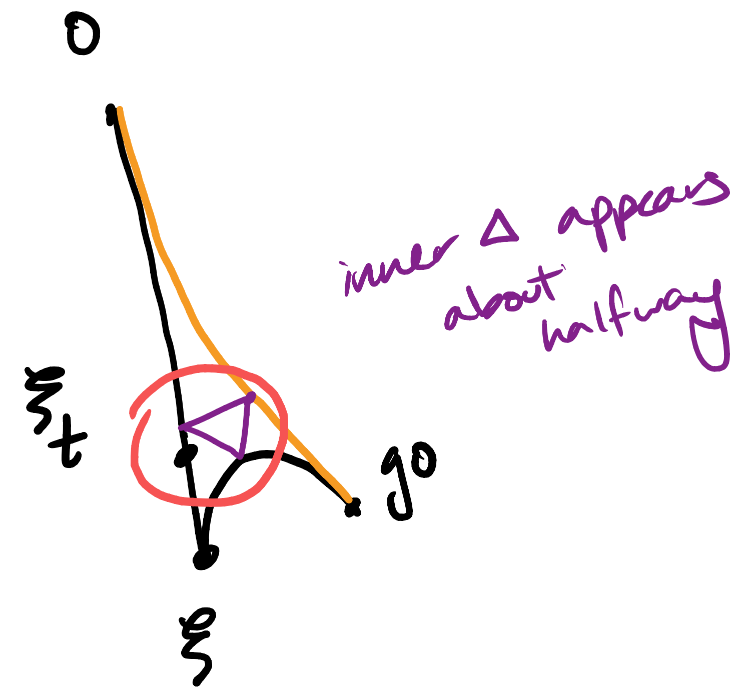
Key Lemma

$$\bigcup_{\substack{g \in \Pi \\ d(o, go) \geq 2t}} gK \text{ "}\sim\text{" } \mathcal{O}_r(o, \xi_t)$$

$$\bigcup_{\substack{g\in\Pi \\ d(o,go)\geq 2t}} gK \text{ "}\sim\text{" } \mathcal{O}_r(o,\xi_t)$$

it is super technical so let's just understand & appreciate the statement





$$\eta \in \mathcal{O}_r(o,\xi_t)$$

$$\Rightarrow \quad d(o,go) \geq 2t$$

density lemma $\to$ $\mu(\mathcal{O}_r(x,\xi_t)) \asymp \sum_{\substack{g\in\Pi \\ d(o,go)\approx 2t}} e^{-\delta_\Gamma d(o,go) + \delta_\Gamma t}$

choose K cpt fundamental domain for $\Pi \curvearrowright \Lambda_\Gamma \setminus \{\xi\}$
by geom. finiteness

$$\mu(\mathcal{O}_r(x,\xi_t)) \asymp \sum_{\substack{g\in\Pi \\ d(o,go)\approx 2t}} \mu(gK)$$

by key lemma

note: no atoms!

$$\frac{d\, g_* \mu_{\xi_t}}{d\mu_{\xi_t}}(\eta) = \frac{d\mu_{g\xi_t}}{d\mu_{\xi_t}}(\eta) \asymp e^{-\delta_\Gamma \beta_\eta(g\xi_t, \xi_t)}$$

<u>coarse hyp lemma</u>: $\forall\, K \subseteq \Lambda_\Gamma \setminus \{\xi\}$ cmpt, $\beta_\xi(go, o)$ large,

$$\beta_\eta(\xi_t, g\xi_t) \asymp d(o, go) + 2t \qquad \forall\, g\xi = \xi \text{ parab.}$$

density lemma follows $\square$

Observe: density lemma is where $B_\Pi(t)$ affects the measure in a significant way.

Talk 4

The Khinchin Theorem

quasi-independence

The logarithm law

Thm (B.-Tiozzo) $\Gamma \curvearrowright (X, d)$ g.f.

[Khinchin-type theorem] $\boxed{\text{one } \Pi \text{ mixed growth, } \delta_\Pi < \delta_\Gamma}$

μ PS measure then

for any Khinchine function φ,

(1) $\mu(\Theta_\lambda(\varphi)) = 0$ if $K_\lambda(\varphi) < \infty$

(2) $\mu(\Theta_\lambda(\varphi)) = 1$ if $K_\lambda(\varphi) = \infty$.

Define for $\lambda < 1$

$$P_n = \{ p \in \Lambda_\Gamma \text{ parabolic s.t. } r_p \in [\lambda^{n+1}, \lambda^n) \}$$

$$S_n = \bigcup_{p \in P_n(\lambda)} H_p(r_p \varphi(r_p))$$

pictures

$$\Theta(\varphi) = \limsup S_n(\lambda)$$

$$= \bigcap_{n=0}^{\infty} \bigcup_{m \geq n} \bigcup_{p \in P_m(\lambda)} H_p(r_p \varphi(r_p))$$

Let $b(t) = (-2\log t + 1)^{a_\Pi}$. then

$$K(\varphi) = \sum_{n \in \mathbb{N}} \varphi(\lambda^n)^{2(\delta_\Gamma - \delta_\Pi)} \, b(\varphi(\lambda^n))$$

Defn: $(Y, \mathbb{P})$ measure space $A_n \subseteq Y$ measurable are quasi-indep. if $\exists\, c$ s.t. $\forall n \neq m$,

$$\mathbb{P}(A_n \cap A_m) \leq c\, \mathbb{P}(A_n)\, \mathbb{P}(A_m)$$

e.g. coin-tosses are exactly independent

A_n = flip heads at step n

A_m = flip heads at step m

Borel - Cantelli Lemma

① $\sum \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup A_n) = 0$

② $\sum \mathbb{P}(A_n) = \infty$ and quasi-independence

$\Rightarrow \mathbb{P}(\limsup A_n) > 0$.

Quasi-independence Lemma

(B-T) The family $\{S_n(\lambda)\}$ is quasi-independent.

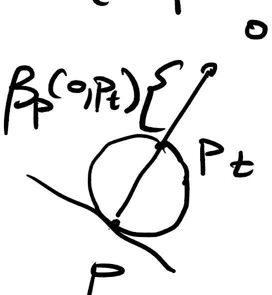
Horoball shadow lemma: Fix θ,

$$\mu(\mathcal{H}_p(\theta r_p)) \asymp \theta^{2(\delta - \delta_\pi)} r_p^{\delta} \, b(\theta)$$

Pf: p_t on geod (o, p)

$$\mu(\mathcal{O}_r(x, p_t)) \asymp$$

$$e^{-\delta_p t} e^{(2\delta_\pi - \delta_p) d(p_t, \Gamma_o)} b(d(p_t, \Gamma_o))$$



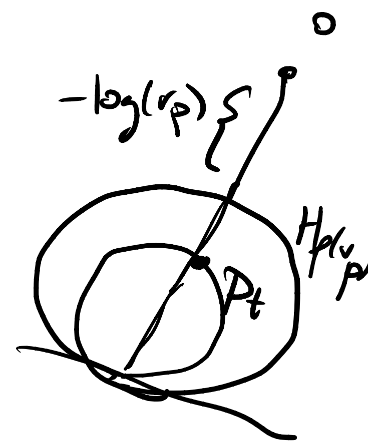
Let $t = -\log \theta r_p$.

$$\text{"}\mathcal{O}_r(x, p_t)\text{"} \asymp \text{"}\mathcal{H}_p(t)\text{"}$$

$$e^{-\delta_p t} = \theta^{\delta} r_p^{\delta}$$

$$d(p_t, \Gamma_o) \asymp -\log \theta$$

$\square$



Note: $\theta = 1$

Defn: $\mathcal{H}_p(\varphi) = \mathcal{H}_p(r_p \varphi(r_p^{-1}))$

$\mathcal{H}_p = \mathcal{H}_p(1)$

Cor 1:

$p \in \mathcal{P}_n(\lambda)$,

$$\mu(\mathcal{H}_p \varphi) \asymp \lambda^{n\delta} \varphi(\lambda^n)^{2(\delta_p - \delta_\pi)} \times b(\varphi(\lambda^n))$$

all about the same size

Lemmee 2

*picture

$$\mu(S_n) \asymp \sum_{p \in \mathcal{P}_n} \mu(\mathcal{H}_p(\varphi))$$

$$\asymp \#\mathcal{P}_n \, \mu(\mathcal{H}_p(\varphi))$$

Yang $\quad \#S_n = \#\mathcal{P}_n \asymp \lambda^{-\delta n}$

Lemma 3 For any $p \in \mathcal{P}_n$,

$$\mu(S_n) \asymp \frac{\mu(\mathcal{H}_p \varphi)}{\mu(\mathcal{H}_p)}$$

$\varphi = 1$

Pf: Lem 2, Yang, Cor 1 for φ and $\varphi = 1$

Now, assume $S_n \cap S_m \neq \emptyset$.

Fix $p^* \in P_n(\lambda)$

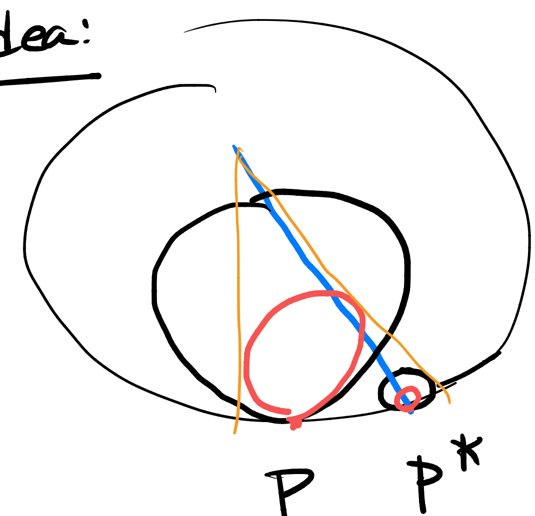
Let $I(p^*) = \{ p \in P_m(\lambda) : \mathcal{H}_p \varphi \cap \mathcal{H}_{p^*} \varphi \neq \emptyset \}$

WLOG $m > n$

Lemma 40

$$\mathcal{H}_p \text{ "}\underset{\sim}{\subset}\text{" } \mathcal{H}_{p^*} \varphi$$

idea:



Cor 2:

$$\# I(p^*) \leq \frac{\mu(\mathcal{H}_{p^*} \varphi)}{\mu(\mathcal{H}_p)}$$

Pf of quasi-independence

$$\mu(S_n \cap S_m)$$

$$\leq \sum_{p^* \in \mathcal{P}_n} \sum_{p \in I(p^*)} \mu(\mathcal{H}_p(\varphi)) \qquad \text{subadditivity}$$

$$\leq \sum_{p^* \in \mathcal{P}_n} \# I(p^*)\, \mu(\mathcal{H}_p(\varphi)) \qquad \text{Cor 1}$$

$$\lesssim \sum_{p^* \in \mathcal{P}_n} \frac{\mu(\mathcal{H}_{p^*}\varphi)}{\mu(\mathcal{H}_p)}\, \mu(\mathcal{H}_p(\varphi)) \qquad p \in \mathcal{P}_m \qquad \text{Cor 2}$$

$$= \Big(\underbrace{\sum_{p^* \in \mathcal{P}_n} \mu(\mathcal{H}_{p^*}\varphi)}_{\text{Lemma 2}} \Big) \underbrace{\frac{\mu(\mathcal{H}_p(\varphi))}{\mu(\mathcal{H}_p)}}_{\text{Lemma 3}} \lesssim \mu(S_n)\,\mu(S_m)$$

$\square$

Pf: of Khinchin-type thm

First, Lemma 1 $\Rightarrow$

$$K(\varphi) \asymp \sum_{n \in \mathbb{N}} \mu(S_n)$$

Since $\Theta(\varphi) = \limsup S_n$

Borel-Cantelli ① $\Rightarrow$ Khinchin theorem (1)

Conversely, Borel-Cantelli ② + quasi-independence lemma

$\Rightarrow \mu(\Theta_\lambda(\varphi)) > 0.$

To prove $\mu(\Theta(\varphi)) = 1$, by ergodicity of μ wrt Γ, it suffices to show $\forall g \in \Gamma$, $\Theta(\varphi)$ is almost Γ-inv.

meaning: $\mu(g\,\Theta(\varphi) \,\Delta\, \Theta(\varphi)) = 0$

i.e. $\mu(g\,\Theta(\varphi)) = \mu(\Theta(\varphi))$

by

Fact: μ ergodic $\Rightarrow$ almost Γ-inv sets have trivial measure.

Note: $\Leftarrow$ is obvious

A Γ-inv $\Rightarrow$

$\mu(\gamma A) = \mu(A)$

Black box $\Theta(\varphi)$ is almost Γ-inv.

$\square$

Logarithm Law

final goal: why we stopped at mixed exp. and everyone else stopped at exp.

"prove" log law, don't verify the limsup exactly, just identify the tn seq & compute the limit.

Thm (B.-Tiozzo)

μ a.e. $\xi \in \Lambda_\rho$,

$$\limsup_{t \to \infty} \frac{d(\xi_t, \Gamma_o)}{\log t} = \frac{1}{2(\delta_\rho - \delta_\pi)}$$

understand where these come from also

"pf" of logarithm law $\boxed{\# \Pi_i = 1}$

Let

to cancel out in $K(\varphi)$

$$\varphi_\varepsilon(x) := \log(x^{-1})^{-\frac{1+\varepsilon}{2(\delta_\Gamma - \delta_\Pi)}}$$

exercise φ_ε is Khinchine

$$K_\lambda(\varphi_\varepsilon) = \sum \varphi_\varepsilon(\lambda^n)^{2(\delta - \delta_\Pi)} \quad (-2\log \varphi_\varepsilon(\lambda^n))^{a_\Pi}$$

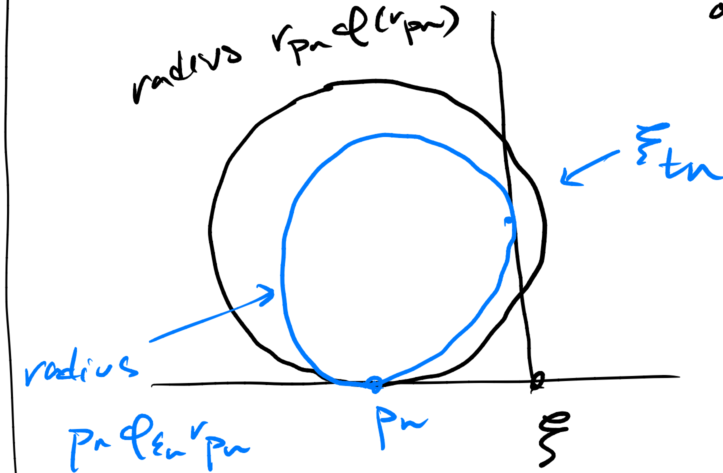
$$= \sum \log(\lambda^{-n})^{-1+\varepsilon} \cdot (-2)^{a_\Pi} \left(\log\left((\log \lambda^{-n})^{1+\varepsilon/2(\delta-\delta_\Pi)}+1\right)\right)^{a_\Pi}$$

$$\asymp \sum \frac{1}{n^{1+\varepsilon}} \log(n+1)^{a_\Pi} \qquad a_\Pi \geq 0$$

calculus exercise ↑ diverges if $\varepsilon = 0$ and converges if $\varepsilon > 0$.

Then μ a.e. $\xi \in \Theta_\lambda(\varphi_0)$, choose maximal seq. $p_n \in P$ so that geodesic $[o,\xi)$ passes through $H_{p_n}(r_{p_n}\varphi_0 r_{p_n})$ in order, and $r_{p_n}\varphi_0 r_{p_n}$ monotone decr.

then, $\exists\, \varepsilon_n > 0$ s.t. $[o,\xi)$ tangent to $H_{p_n}(r_{p_n}\varphi_{\varepsilon_n} r_{p_n})$. ξ_{t_n} pt of tangency on $[o,\xi)$.



claim:

$$\limsup_{t\to+\infty} \frac{d(\xi_t, \Gamma o)}{\log t} = \limsup_{n\to\infty} \frac{d(\xi_{t_n}, \Gamma o)}{\log t_n}$$

idea: $d(\xi_{t_n}, \Gamma o) \approx$ distance between horospheres maximized at $t = t_n$

Technical hyperbolicity lemmas

$\rightsquigarrow$

$$d(\xi_{t_n}, \Gamma_0) \underset{\sim}{\approx} -\log(\varphi_{\varepsilon_n}(r_{p_n}))$$

not really but close enough

more on this

$$\text{"}\sim\text{"} \; \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\pi)} \log t_n$$

If $\varepsilon_n > \varepsilon > 0$ some ε, then

$\xi \in \Theta_\lambda(\varphi_\varepsilon)$. But $\mu(\Theta_\lambda(\varphi_\varepsilon)) = 0$,

so we can choose ξ s.t.

$\xi \in \Theta_\lambda(\varphi_0)$ but $\xi \notin \Theta_\lambda(\varphi_\varepsilon)$

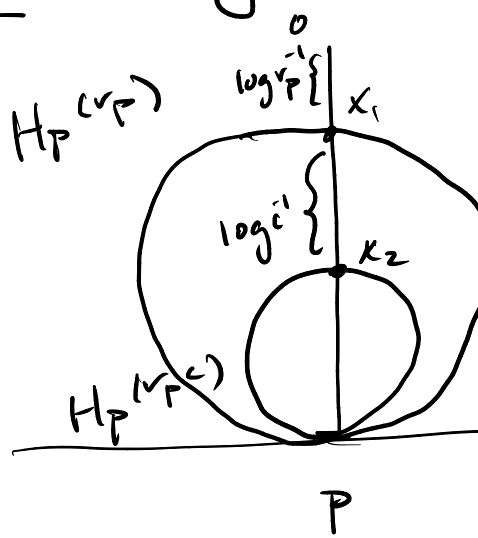
$\forall \varepsilon > 0$. Thus, $\varepsilon_n \to 0$ and

$$\limsup = \frac{1}{2(\delta_\Gamma - \delta_\pi)}$$

Lemma 1:

$$d(\xi_{t_n}, \Gamma_0) \approx -\log(\varphi_{\varepsilon_n}(r_{p_n}))$$

<u>idea:</u> for any $c<1$



cocompactness $\Rightarrow$ Γ_0 bndd dist from $H_p(r_p)$.

How much further to get to $H_p(r_p c)$?

$$\beta_p(o, x_1) = -\log r_p$$

$$\beta_p(o, x_2) = -\log r_p c$$

intuition: horospheres are "balls centered at p", x_1, x_2 colinear on geod to p

cocycle property

$$\beta_p(x_1, o) + \beta_p(o, x_2) = \beta_p(x_1, x_2) = d(x_1, x_2)$$

$$\| \quad \log r_p - \log r_p - \log c = -\log c.$$

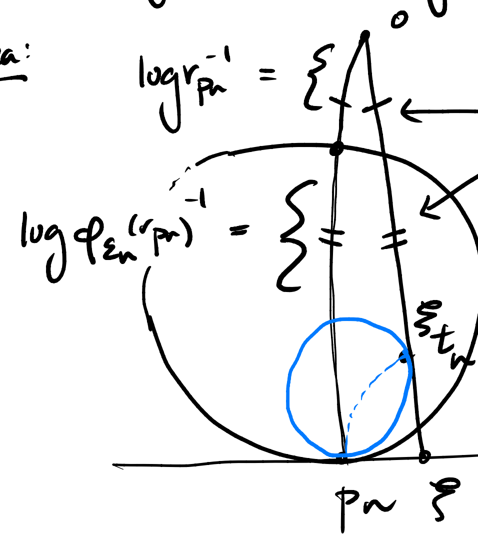
Now take $c = \varphi_\varepsilon r_{p_n}$ □

Lemma 2:

additive consts

$$t_n + \log \varphi_{\varepsilon_n}(r_{p_n}) \lesssim \log r_{p_n}^{-1} \lesssim t_n$$

<u>idea:</u>



hyp metric space lemma: ξ_{t_n} = closest point projection of p_n to $[o, \xi)$ = bndd dist from inner triangle □

(L1)

$$\Rightarrow d(\xi_{t_n}, \Gamma_0) \sim -\log \varphi_{\varepsilon_n} r_{p_n}$$

$$= -\log\left((\log r_{p_n}^{-1})^{-(1+\varepsilon_n)/2(\delta_\Gamma - \delta_\Pi)}\right)$$

$$= \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log \log r_{p_n}^{-1}$$

$$\leq \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log t_n \quad (L2)$$

$$\Rightarrow \limsup \leq \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \quad \forall n.$$

If $\varepsilon_n > \varepsilon > 0$ some ε, then

$\xi \in \Theta_\lambda(\varphi_\varepsilon)$. But $\mu(\Theta_\lambda(\varphi_\varepsilon)) = 0$,

so we can choose ξ s.t.

$\xi \in \Theta_\lambda(\varphi_0)$ but $\xi \notin \Theta_\lambda(\varphi_\varepsilon)$

$\forall\, \varepsilon > 0$. Thus, $\varepsilon_n \to 0$ and

the upper bound follows.

more details if desired

Lower bound

$$d(\xi_{t_n}, \Gamma_0) \approx -\log \varphi_{\varepsilon_n}{}' r_{p_n}$$

$$= \frac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log \log r_{p_n}^{-1}$$

(L2) lower $\gtrsim \dfrac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \left(\log\left(t_n - \dfrac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log\log(r_{p_n}^{-1})\right)\right)$

(L2) upper $\gtrsim \dfrac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log\left(t_n - \dfrac{1+\varepsilon_n}{2(\delta_\Gamma - \delta_\Pi)} \log(t_n)\right)$

$\Rightarrow \limsup \geq \dfrac{1}{2(\delta_\Gamma - \delta_\Pi)}$ since $t_n \to \infty$, $\varepsilon_n \to 0$

$\square$